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# Changing Coordinates in the Context of Orbital Mechanics

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**ABSTRACT**

This note works through an example of switching between many coordinate systems using a modern matrix language that lends itself to describing arenas with multiple entities such as found in many Defence scenarios. To this end, it describes an example in planetary orbital theory, whose various Sun- and Earth-centred coordinate systems makes that theory a good test-bed for such an exposition of changing coordinates. In particular, we predict the look direction to Jupiter from a given place on Earth at a given time, highlighting the careful book-keeping that is required along the way. To avoid much of the rather antiquated jargon and notation that pervades orbital theory, we explain the first principles of 2-body orbital motion (Kepler's theory), beginning with Newton's laws and proving all the necessary expressions. The systematic and modern approach to changing coordinates described here can also be applied just as readily in contexts such as a Defence aerospace engagement, which follows the interaction of multiple entities that each carry their own coordinate system.

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# Changing Coordinates in the Context of Orbital Mechanics

## Executive Summary


Real-world defence scenarios might be described or managed by any of their participants, and a core element of this description is the ability to transform between the many coordinate systems that typically quantify the entities involved. Switching between coordinates is often seen as a classical yet difficult problem. This report attempts to show that the task can be made easier and more transparent by using unambiguous notation that carefully describes all relationships of the relevant entities.

A worked example in planetary orbital theory is a useful test-bed for such an exposition. The various Sun- and Earth-centred coordinate systems involved in predicting, say, the look direction to Jupiter from a given place on Earth at a given time require careful book-keeping of the plethora of numbers involved in the calculation.

With that worked example in our sights, and to avoid much of the rather antiquated jargon and notation that pervades orbital theory, we cover the first principles of 2-body orbital motion by beginning with Newton's laws and proving all the necessary expressions. The main focus here is to show how to interrelate the various coordinate systems that are necessary to the worked example.

We are content to consider the 2-body problem (Kepler's theory)—which can be solved analytically, unlike the many-body problem—because the relevant concepts of changing coordinates are sufficiently illustrated in a 2-body scenario. We thus decouple the Solar System into two 2-body systems that are gravitationally independent: Sun–Earth, and Sun–Jupiter. The resulting high accuracy in the prediction of Jupiter's look direction from Earth supports the validity of this decoupling.

The exposition begins with the relevant classical mechanics and time concepts, proves Kepler's three laws, then establishes and describes how to relate the different coordinate systems involved with the Earth-centred and Sun-centred inertial frames, the Earth-centred Earth-fixed frame, and the observer's local "flat Earth" frame. It describes the necessary celestial geometry and orbital elements, and finishes with the worked example of locating Jupiter at a given time.

The explanations in the pages that follow are written in an expansive style that describes related concepts, such as what equinoxes and solstices are and when they occur, how julian days are defined, and how orbital calculations can be extended to more complex motion, such as that of the Moon. In particular, the theory of how to change coordinates more generally in an elegant but also powerful way is explained in detail.

Although predicting where Jupiter can be found is not of any great utility in a Defence context, anyone who understands the procedure described in this note will have no problem attempting the simpler task of working, for example, in the area of research into satellite positioning systems such as GPS.

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# Contents

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# 1 Introduction

This note serves two purposes. First, it gives a worked example in modern mathematical language of the subject of coordinate transforms, a subject that is the acknowledged bane of many a researcher in Defence, where knowledge is sought of the interaction of many entities with each using its own coordinate system to communicate its view of a scenario. Second, this report describes orbital theory in a streamlined way that is perhaps more in line with a modern physicist's way of viewing the subject than is traditional in the subject, while at the same time avoiding extraneous concepts such as the theory of interplanetary trajectories.

The world's increasing reliance on satellite technology for timing and positioning suggests a need for more awareness of the dynamics of Earth satellites. For example, in the study of global positioning systems such knowledge enables a researcher to construct and explore an orbit in an informed way, and to understand the relations between equivalent choices of parameters that describe the orbit.

The sections that follow cover the relevant classical mechanics and time concepts from the ground up. We establish the necessary frames and coordinates, and describe how to relate the different coordinate systems that are required to calculate a satellite's position. We opt to work through a particularly complicated example: finding the bearing and elevation of the planet Jupiter in an Adelaide sky at a given time. Although Jupiter is hardly a satellite of Earth, the method of predicting its position covers more ground than that of forecasting the position of an Earth satellite, and so serves as a more in-depth study of the general concepts. The sections that follow also segue into descriptions of related concepts, such as details of Earth's orbit and how it relates to the seasons, and the "julian day" approach to measuring time.

# 2 Orbits from Newton's Laws

Orbital mechanics is an old and established subject. Some of its language was first coined hundreds of years ago, and the subject is generally still presented in ways that reflect its mediaeval beginnings. Its literature can often use a style that most physicists (myself included) will probably regard as obscure: for example, aside from sometimes archaic terminology, even modern books might treat the Sun as orbiting Earth, with the planets orbiting the Sun [1]. This choice of (non-inertial) frame is useful qualitatively and can be argued for and against from a philosophical point of view, but it lacks the simplicity of an inertial frame for calculating, and has something of the mediaeval about it. It is rare to find a book on the subject that presents succinctly all that is needed to locate a heavenly body without being merely a "cook book" and also without presenting much extraneous information at the same time [2]; most books cover a lot of ground over many chapters, without always getting to the heart of the subject quickly. The treatment of vectors and coordinate systems in many books can benefit from following the more modern approach used in [3] that is specifically tailored to cope easily with multiple coordinate systems.

The question we will answer in this report is the following: if you gaze up at the sky tonight at 8 p.m., where must you look to find a given planet? Making such a prediction at a given time and place involves a wealth of physics and mathematics, and seeing the calculations come together is nothing less than doing an experiment in classical mechanics on a grand scale. In this report I'll show precisely how to do that calculation, by building the necessary theory

from first principles and then joining this to a set of published orbital elements describing the planet's orbit that have been produced from astronomers' observations.

Predicting where a planet will be seen at some time and place follows time-honoured classical mechanics. First use "force equals mass times acceleration" to analyse a situation governed fully by gravity, to find the planet's position in its orbit as a function of time, using appropriate initial conditions. Orientate this orbit (together with the planet) correctly relative to the Sun, do the same for Earth, and sight the planet from Earth. Finally, express the resulting vector as a bearing and elevation for the given place of observation on Earth.

Modern fast computers calculate planetary ephemerides (tables of predicted positions over time) by incrementally solving many differential equations for the $n$-body problem that combine the many gravitational influences on and due to the planets. We would have no choice but to calculate this way if we required highly accurate predictions. In contrast, treating the chosen planet–Sun pair as a 2-body problem in gravity makes it exactly solvable, and the result is sufficiently simple and accurate to give an appreciation for the physics involved. The resulting *keplerian orbit* forms the basis of all orbital theory. We will make the approximation that for the purpose of predicting Jupiter's direction from Earth, the Solar System's dynamics can be decoupled into two 2-body systems that are gravitationally independent: Sun–Earth, and Sun–Jupiter. Each of these 2-body problems can then be solved individually. This approximation works extremely well in practice. Additionally, the 2-body treatment is completely sufficient to serve as a platform for discussing the necessary coordinate transforms.

In the 2-body problem, the planet is subject to only the gravitational force exerted by the Sun. It's an undergraduate task in mechanics to show that the motions of the centres of mass of these assumed-spherical bodies is identical to the motions calculated by replacing the planet and Sun with two point masses that are located at the centres of the spheres and have those spheres' masses. We'll assume this result.

Now consider such a "point" planet of mass $m$ interacting with its primary (say, the Sun) of mass $M$, and measure the planet's position $\boldsymbol{r}$ relative to its primary. The "central force" exerted by the Sun implies that the planet moves in a plane, and so we place the Sun at the origin of *orbit-plane* coordinates, a cartesian set $x_{\text{OP}}, y_{\text{OP}}, z_{\text{OP}}$. (These are more conventionally called *perifocal*, or *PQW* coordinates). Begin by analysing the planet's motion in the orbit plane $x_{\text{OP}}\, y_{\text{OP}}$ using polar coordinates $r, \theta$, where $r \equiv |\boldsymbol{r}|$, as shown in Figure 1.

Every point in the orbit plane has attached to it two unit vectors: the radial unit-length vector $\boldsymbol{u}_r$ points everywhere radially outward from the Sun, and the transverse unit-length vector $\boldsymbol{u}_\theta$ is produced by rotating $\boldsymbol{u}_r$ by 90° in the orbit plane right-handed around $z_{\text{OP}}$. Referring to Figure 1, we apply Newton's law $\boldsymbol{F} = m\boldsymbol{a}$ to the mass-$m$ planet:

$$\frac{-GMm}{r^2}\boldsymbol{u}_r = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{bmatrix}\text{position of } m \text{ relative} \\ \text{to } \textit{any} \text{ point fixed in} \\ \textit{any} \text{ inertial frame}\end{bmatrix}, \tag{2.1}$$

where $G$ is Newton's gravitational constant. A frame is defined to be inertial if a mass released at rest in that frame remains motionless indefinitely. Any frame moving at constant velocity relative to an inertial frame will itself be inertial. A convenient choice of inertial frame for our scenario is the planet–primary centre-of-mass frame (it can be shown that centre-of-mass frames are inertial), and a convenient point fixed to this frame is the centre of mass itself. With $r$ the length of the planet's position vector $\boldsymbol{r} = r\boldsymbol{u}_r$ extending from its primary, (2.1)

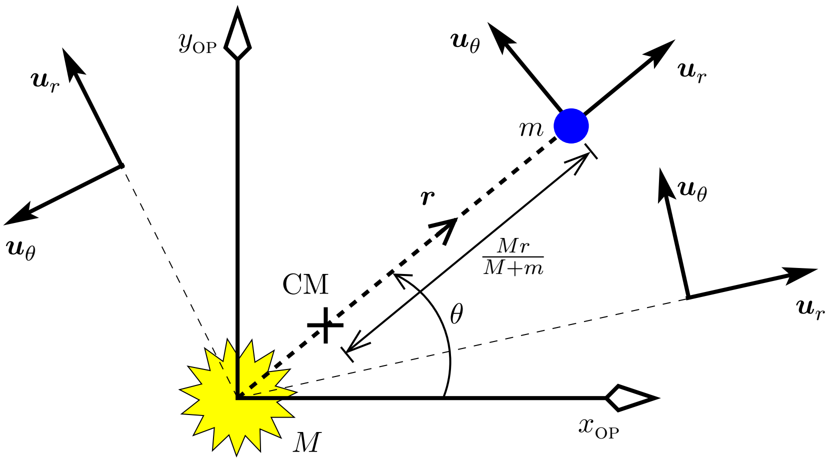



*Figure 1: Arrangement of vectors and masses for deriving the equation of motion of a satellite of mass $m$ orbiting a primary body of mass $M$. "CM" denotes the centre of mass of $M$ and $m$: it divides the line joining the masses in the ratio $m : M$. Unit-length basis vectors at three representative points are shown.*

becomes (see [4] for a side comment)

$$\frac{-GMm}{r^2}\boldsymbol{u}_r = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{M\boldsymbol{r}}{M+m} = \frac{Mm}{M+m}\ddot{\boldsymbol{r}} \tag{2.2}$$

(where the dots denote two time differentiations), which rearranges to

$$\ddot{\boldsymbol{r}} = \frac{-G(M+m)}{r^2}\boldsymbol{u}_r \equiv \frac{-\mu}{r^2}\boldsymbol{u}_r\,, \tag{2.3}$$

where $\mu$ is conventional shorthand for $G(M+m)$. If the above analysis seems straightforward, realise that we have derived the basic equation to be solved, (2.3), in a very economical way, due to our combining the Sun-origin coordinates with the "CM-at-rest" inertial frame.

We will solve (2.3) by expressing $\ddot{\boldsymbol{r}}$ in terms of $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$, and then equating components of these across the equals sign in (2.3). To begin doing so, we establish a terminology that will become important later, by distinguishing notationally between a *proper vector* (an arrow such as $\boldsymbol{r}$), and its description in some coordinate system $A$, being an array of coordinates $[\boldsymbol{r}]_A$ known as a *coordinate vector* [5]. So write $\boldsymbol{r}$ in the orbit plane's cartesian coordinates, indicated by $[\boldsymbol{r}]_{\mathrm{OP}}$:

$$[\boldsymbol{r}]_{\mathrm{OP}} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \equiv \begin{bmatrix} rc \\ rs \end{bmatrix}. \tag{2.4}$$

Differentiating a vector is easy when using cartesian coordinates: just differentiate its components [6]. So twice differentiating each element of (2.4) gives

$$[\ddot{\boldsymbol{r}}]_{\mathrm{OP}} = \begin{bmatrix} (\ddot{r} - r\dot{\theta}^2)c - (2\dot{r}\dot{\theta} + r\ddot{\theta})s \\ (\ddot{r} - r\dot{\theta}^2)s + (2\dot{r}\dot{\theta} + r\ddot{\theta})c \end{bmatrix}. \tag{2.5}$$

Now realise that

$$[\boldsymbol{u}_r]_{\text{OP}} = \begin{bmatrix} c \\ s \end{bmatrix}, \quad [\boldsymbol{u}_\theta]_{\text{OP}} = \begin{bmatrix} -s \\ c \end{bmatrix}, \tag{2.6}$$

and combining these with (2.5) enables $\ddot{\boldsymbol{r}}$ to be expressed without using cartesian coordinates:[1]

$$\ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{u}_\theta\,. \tag{2.7}$$

Comparing (2.3) with (2.7) produces the two key equations of the planet's motion [8]:

$$\ddot{r} - r\dot{\theta}^2 = -\mu/r^2\,, \qquad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0\,. \tag{2.8}$$

The second equation in (2.8) can be rewritten as

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0\,, \tag{2.9}$$

or $r^2\dot{\theta}$ = a constant, conventionally called $h$. Note that for a planet of mass $m$ and velocity $\boldsymbol{v}$ relative to $M$, the angular momentum vector per unit mass, relative to $M$, is

$$\boldsymbol{L}/m = \boldsymbol{r} \times \boldsymbol{v} = \boldsymbol{r} \times \dot{\boldsymbol{r}}\,. \tag{2.10}$$

Remember that $\boldsymbol{r} = r\boldsymbol{u}_r$, and just as we found $\ddot{\boldsymbol{r}}$ above, we can also differentiate each element of (2.4) just once, and again use (2.6) to produce

$$\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta\,. \tag{2.11}$$

Substitute these expressions for $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ into (2.10), and apply the distributive law to the cross product to give

$$\boldsymbol{L}/m = r\boldsymbol{u}_r \times (\dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta) = r^2\dot{\theta}\boldsymbol{u}_{z_{\text{OP}}} = h\boldsymbol{u}_{z_{\text{OP}}}\,, \tag{2.12}$$

where $\boldsymbol{u}_{z_{\text{OP}}}$ is the unit vector normal to the orbit plane. So $h$ is the planet's orbital angular momentum per unit mass with respect to its primary. Remember that because the planet is conventionally chosen to orbit in the direction of increasing $\theta$, the angular momentum $\boldsymbol{L}$ will be parallel to $\boldsymbol{u}_{z_{\text{OP}}}$, making $h$ always positive. We'll use the fact that $h$ is the $z_{\text{OP}}$ component of $\boldsymbol{r} \times \boldsymbol{v}$ ahead in (4.3).

Now solve the first equation in (2.8) by a change of variables $u = 1/r$, eliminating $t$ in favour of $\theta$. To do this, note that $t$ can be treated as a function of $\theta$ alone, in which case the chain rule of differentiation combines with the definition of $h$ to give $\mathrm{d}/\mathrm{d}t = \mathrm{d}\theta/\mathrm{d}t \times \mathrm{d}/\mathrm{d}\theta = hu^2\,\mathrm{d}/\mathrm{d}\theta$. So we calculate $\ddot{r}$ by applying two time derivatives of $hu^2\,\mathrm{d}/\mathrm{d}\theta$ to $1/u$. The result is

$$\ddot{r} = -h^2u^2\,\mathrm{d}^2u/\mathrm{d}\theta^2\,. \tag{2.13}$$

The first equation in (2.8) becomes

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{\mu}{h^2}\,. \tag{2.14}$$

This linear differential equation is easily solved to give $u = B\cos(\theta - \theta_0) + \mu/h^2$ for constants $B$ and $\theta_0$, and from this we can write

$$r = \frac{1}{B\cos(\theta - \theta_0) + \mu/h^2} \equiv \frac{p}{1 + e\cos(\theta - \theta_0)}\,, \tag{2.15}$$

where $p \equiv h^2/\mu$ is called the orbit's *semi-parameter* and $e$ its *eccentricity*. The angle $\theta_0$ merely specifies some bulk rotation of the orbit, so we will set it to zero without loss of generality.

[1] See [7] for an alternative way of producing (2.7), that doesn't rely on cartesian coordinates.

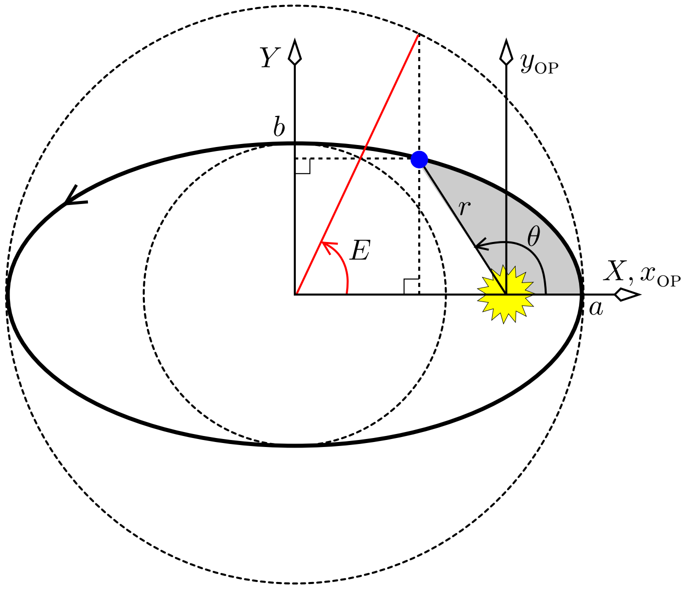



*Figure 2: Basic angles and axes describing an elliptical orbit. The dotted circles aid in visualising how the eccentric anomaly E is defined on page 7.*

# 3 Kepler's Three Laws

When $0 \leqslant e < 1$, (2.15) with $\theta_0 = 0$ describes an ellipse in the $x_{OP}\, y_{OP}$ plane symmetrical about the $x_{OP}$ axis, shown in Figure 2. This is the path followed by a planet with insufficient energy to escape its primary, since values of $e \geqslant 1$ give parabolae and hyperbolae, open conic sections that describe only unbound objects such as comets.

The orbital ellipse has semi-major and semi-minor axes lengths $a$ and $b$. Its centre of symmetry is the origin of the $XY$ axes in Figure 2, so that it satisfies $X^2/a^2 + Y^2/b^2 = 1$. The Sun occupies one focus, at the origin of the $x_{OP}\, y_{OP}$ axes where $X = ae$. The semi-parameter $p$ is the value of $r$ when $\theta = 90°$, and Pythagoras's theorem produces $p = a(1 - e^2)$.

Setting $\theta_0$ to zero corresponds to measuring $\theta$ from the planet's *perifocus*: its point of closest approach to the focus. (The perifocus is also called *perihelion* for orbits about the Sun, and *perigee* for orbits about Earth.) This setting of $\theta_0$ to zero is always done, in which case $\theta$ is called the planet's *true anomaly*. Here, the word "anomaly" isn't meant to imply that something is wrong; perhaps it originally meant a *departure* from perifocus. One theory suggests that the language of orbital mechanics was deliberately obfuscated by early navigators to prevent widespread knowledge of how to navigate, thus discouraging would-be mutineers [9].

We've found that the orbit of a planet is an ellipse with the Sun at one focus: this is Kepler's First Law. In general (2.15) describes a conic section—and only a conic section. A sometimes-found misconception holds that if momentarily perturbed, a planet's orbit would collapse and send it spiralling in to the Sun. But this cannot be; a spiral path is not a conic section, so cannot describe planetary motion. No matter how slowly it moves, a *point* mass $m$ that is set to move not purely radially can never fall in to collide with a *point* mass $M$; *all* motion of bodies under gravity is a part of an unending orbital motion, unless the non-pointlike nature of the bodies gets in the way. When you throw a rock, then neglecting air resistance, from the moment it leaves your hand until it hits the ground, it is orbiting Earth's

centre in a tight ellipse. The end of the ellipse along which the rock flies before it hits the ground is modelled very accurately as a parabola, but the full path is an ellipse. (Of course, Earth satellites *do* eventually fall to the ground if they orbit low enough, due to atmospheric drag that imposes a force which we have not included above.)

Related to this idea of an orbiting rock is the term "orbital speed"; for example, the orbital speed of a satellite of Earth is generally given as 8 km/s. How does this relate to the rock a few lines up? Although a thrown rock is in orbit, its elliptical orbit intersects Earth's surface, so it will quickly hit the ground. The orbital speed of a rock is something of an idealisation, being the (constant) speed that the rock needs in order to orbit Earth in the smallest *circle* that never touches the ground.

Figure 2 includes a shaded area $A$ that is swept out by the Sun-to-planet vector $\boldsymbol{r}$ beginning at perihelion. Simple geometry shows that the infinitesimal orbital area $\mathrm{d}A$ swept out by this vector in a time interval $\mathrm{d}t$ is $\mathrm{d}A = r^2\mathrm{d}\theta/2$. It follows that

$$\mathrm{d}A/\mathrm{d}t = r^2\dot{\theta}/2 = h/2\,, \tag{3.1}$$

which is a constant. So the planet's position vector sweeps out area at a constant rate: this is Kepler's Second Law. Now integrate (3.1) over one period $T$ of the planet, noting that the area then swept by the planet is the area $\pi ab$ of its elliptical orbit:

$$\pi ab = \int_0^{\pi ab} \mathrm{d}A = \int_0^T \frac{\mathrm{d}A}{\mathrm{d}t}\,\mathrm{d}t = \int_0^T \frac{h}{2}\,\mathrm{d}t = \frac{hT}{2}\,. \tag{3.2}$$

It follows that $T = 2\pi ab/h$. Squaring this equation and eliminating $b$ and $h$ by using $b = a\sqrt{1-e^2}$ and $h = \sqrt{\mu p} = \sqrt{\mu a(1-e^2)}$ produces Kepler's Third Law:

$$T^2 = \frac{4\pi^2}{G(M+m)}a^3\,. \tag{3.3}$$

Each planet's mass is much less than that of the Sun, so (3.3) becomes $T^2 \propto a^3$ with a constant of proportionality that is approximately the same for all the planets, as found empirically by Kepler. We've extracted Kepler's three laws from Newton's theory of gravity, and indeed this was *the* major early success of, and support for, Newton's work. Kepler is often seen as having lived in the mediaeval era "BN" ("Before Newton"), but Kepler himself came very close to deducing the law of gravitation and creating the subject of calculus, and it was very much partly on his shoulders that Newton stood. In fact, Newton was criticised by his contemporaries for not giving Kepler his due in this regard [10].

# 4 Introducing a Time Dependence

Equation (3.1) re-introduced time into the description of the planet's orbit, after we had temporarily replaced $t$ by $\theta$ just before (2.13). The increase in swept area with time is traditionally described in more detail by introducing a new angle. Because the ellipse satisfies $X^2/a^2 + Y^2/b^2 = 1$, this new angle $E$ can be defined such that

$$\sin E \equiv \frac{Y}{b} = \frac{r\sin\theta}{b}\,, \quad \cos E \equiv \frac{X}{a} = \frac{ae + r\cos\theta}{a}\,. \tag{4.1}$$

The parameter $E$ is called the planet's *eccentric anomaly*, and is the angle from the $X$ axis to the red line drawn from the $XY$ origin in Figure 2. A few lines of algebra yield $r = a(1 - e\cos E)$, which together with (4.1) means that from $E$ we can extract the planet's polar coordinates $(r, \theta)$. But how does $E$ depend on time? Recall the usual relations $x_{\text{OP}} = r\cos\theta$ and $y_{\text{OP}} = r\sin\theta$; then with $E$ expressed in radians, it follows that

$$\begin{aligned} x_{\text{OP}} &= a\cos E - ae\,, & y_{\text{OP}} &= b\sin E\,, \\ \dot{x}_{\text{OP}} &= -a\dot{E}\sin E\,, & \dot{y}_{\text{OP}} &= b\dot{E}\cos E\,. \end{aligned} \tag{4.2}$$

Now remember from (2.12) that $h$ is the $z_{\text{OP}}$ component of $\boldsymbol{r} \times \boldsymbol{v}$; also, because in the full three-dimensional OP coordinates we can write

$$\begin{aligned} [\boldsymbol{r} \times \boldsymbol{v}]_{\text{OP}} &= (x_{\text{OP}},\ y_{\text{OP}},\ 0) \times (\dot{x}_{\text{OP}},\ \dot{y}_{\text{OP}},\ 0) \\ &= (0,\ 0,\ x_{\text{OP}}\dot{y}_{\text{OP}} - \dot{x}_{\text{OP}}y_{\text{OP}})\,, \end{aligned} \tag{4.3}$$

it follows that $h = x_{\text{OP}}\dot{y}_{\text{OP}} - \dot{x}_{\text{OP}}y_{\text{OP}}$. By substituting from (4.2) into this last expression for $h$, we rewrite $h$ in terms of $E$; then equating with $h = 2\pi ab/T$ from (3.2) produces

$$\dot{E}\,(1 - e\cos E) = 2\pi/T\,, \tag{4.4}$$

which can be written as

$$(1 - e\cos E)\,\mathrm{d}E = 2\pi\,\mathrm{d}t/T\,. \tag{4.5}$$

Integrating this equation from $t = t_{\text{peri}}$ when the planet is at its perifocus (where $\theta$ and $E$ are both zero), we arrive at the celebrated *Kepler's equation* for the time evolution of $E$:

$$E - e\sin E = 2\pi(t - t_{\text{peri}})/T\,. \tag{4.6}$$

(Remember, $E$ must be expressed in radians when using (4.6), even though we are free to write it as a number of degrees when not using that equation.) Kepler's equation tells us how the planet moves along its orbit. The quantity $E - e\sin E$ clearly increases *uniformly* with time from 0 to $2\pi$ over one period, which gives rise to its name the *mean anomaly* $M$:

$$M \equiv E - e\sin E\,. \tag{4.7}$$

Kepler's equation (4.6) can then be written simply as

$$\mathrm{d}M/\mathrm{d}t = 2\pi/T\,. \tag{4.8}$$

All three anomalies—true $\theta$, eccentric $E$, and mean $M$—start at 0 at the perifocus and reach $2\pi$ one period later; but they advance at different rates, and only $M$ advances uniformly, which is precisely why $M$ exists and why it's useful. Unlike $\theta$ and $E$, the mean anomaly $M$ has no geometric interpretation; being simply shorthand for $E - e\sin E$, it is not "natively" an angle any more than $\sin E$ is an angle. But because $M$ increases uniformly from 0 to $2\pi$ over one period, it can be treated as the angle traced by a point orbiting the Sun (or indeed any other centre) at constant speed in a circle. So in that sense the mean anomaly fulfills the dream of the ancients: to reduce a planet's motion to constant speed in one circle! [11]

As a side note on Kepler's Second Law, because both the mean anomaly $M$ and the swept area $A$ increase at constant rates as the planet moves from its perifocus at time zero, the *relative* swept area $A/(\pi ab)$ must equal $M/(2\pi)$.

As an example of solving Kepler's equation, we calculate how far advanced a planet is from perifocus (i.e. the value of $\theta$, the true anomaly) after a quarter period for an orbit of eccentricity $e = 0.1$. The answer is not 90°—that's the *mean* anomaly! Instead, begin by solving (4.6) for $t - t_{\text{peri}} = T/4$, taking care to use radians when solving Kepler's equation:

$$E - 0.1 \sin E = \pi/2\,. \tag{4.9}$$

For ellipses ($e < 1$), Kepler's equation is always easily solved by writing it, or (4.9) in this case, as

$$E = 0.1 \sin E + \pi/2\,, \tag{4.10}$$

which is then iterated from an initial guess of $E$ that suffices always to be zero: that is, begin by setting $E = 0$ in the right-hand side of (4.10) and repeatedly recalculating $E$ using the same equation, inserting the latest value of $E$ into the right-hand side on each iteration. Remember to work in radians! After just 4 iterations here the value of $E$ converges to 1.6703, or about 95.7°. Now (4.1) gives $\sin\theta$ and $\cos\theta$:

$$\sin\theta = \frac{a}{r}\sqrt{1-e^2}\,\sin E \simeq 0.990\frac{a}{r}\,, \quad \cos\theta = \frac{a}{r}(\cos E - e) \simeq -0.199\frac{a}{r}\,. \tag{4.11}$$

The unknown positive number $a/r$ cancels if we calculate $\tan\theta$, and $\theta$ is clearly in the second quadrant. It follows that $\theta = 101.4°$ [12]. To summarise, for this eccentricity of $e = 0.1$, after a quarter period the three anomalies are

$$M = 90°, \quad E \simeq 95.7°, \quad \theta \simeq 101.4°. \tag{4.12}$$

As an aside, we've found the planet's position, but what is its velocity? This velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$ is already known in OP coordinates from (4.2):

$$[\boldsymbol{v}]_{\text{OP}} = (\dot{x}_{\text{OP}},\ \dot{y}_{\text{OP}},\ 0) = \dot{E}\,(-a\sin E,\ b\cos E,\ 0)\,, \tag{4.13}$$

so with $\dot{E}$ given by (4.4), the velocity's OP coordinates are

$$[\boldsymbol{v}]_{\text{OP}} = \frac{2\pi(-a\sin E,\ b\cos E,\ 0)}{T(1-e\cos E)}\,. \tag{4.14}$$

Another useful equation results if we form the dot product of (2.3) with $2\dot{\boldsymbol{r}}$. An integration and some manipulations lead to the *vis-viva equation* that relates the planet's speed $v$ (relative to the Sun) to its current distance $r$ from the primary:

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right). \tag{4.15}$$

The vis-viva equation applies generally to elliptical orbits and doesn't require $M \gg m$. In fact, with an appropriate definition of $a$ it holds for *all* conic-section motion. It's used extensively in the art of manoeuvring spacecraft because it specifies where a craft should switch from one orbit to another, even if one or both of these orbits is unbounded. In the case of a circular orbit and $M \gg m$, it reduces to the well-known $v^2 = GM/r$.

## 4.1 The Use of One Global Time Variable

Knowing how a planet moves, we wish to locate it at some given time; and for this, a single global time variable $t$ is a necessity. We will use Greenwich Mean Time (GMT) [13], but instead of using a 24-hour clock that resets to zero each midnight, we measure $t$ in days (including any fraction) from a date far in the past and never reset it. The number of days elapsed since this remote starting moment is known as the *julian day* (JD)—no relation to the *julian date*, which is simply a date in the julian calendar. The JD is a single real-number way of expressing GMT and so, like GMT, defines a global instant [14].

The remote epoch of julian day zero is 12:00 noon GMT on an arbitrary date chosen far in the past purely so as to render julian days conveniently positive throughout recorded history. The date chosen in the julian scheme was 1st January 4713 BC in the *proleptic julian calendar*, which is an idealised "well-behaved" version of the historical julian calendar—that is, without the sporadic changes in the decreed lengths of certain months that occurred in the historical julian calendar in Roman times. This start moment is related to a confluence of astronomical cycles and the tax/census cycle of ancient Rome, and has no historical significance. In the gregorian calendar, this start moment was 12:00 noon GMT on 24th November 4714 BC [15].

The JD is a real number, so typically is specified to several decimal places. A standard algorithm to convert a "year, month, day, hour, minute, second" to a julian day is the following, in which the "floor" function returns the largest integer less than its argument. Set

$$\begin{aligned} a &= \text{floor}[(14 - \textit{month})/12]\,, \\ y &= \textit{year} + 4800 - a\,, \\ m &= \textit{month} + 12a - 3\,. \end{aligned} \tag{4.16}$$

Then with "$H, M, S$" = hour, minute, second, a date expressed in the (proleptic) *julian* calendar has a julian day of

$$\begin{aligned} \text{JD} = \textit{day} &+ \text{floor}[(153m + 2)/5] + 365y + \text{floor}(y/4) \\ &- 32{,}083 + (H - 12 + M/60 + S/3600)/24\,. \end{aligned} \tag{4.17}$$

A date in the *gregorian* calendar has a julian day of

$$\begin{aligned} \text{JD} = \textit{day} &+ \text{floor}[(153m + 2)/5] + 365y + \text{floor}(y/4) \\ &- \text{floor}(y/100) + \text{floor}(y/400) - 32{,}045 \\ &+ (H - 12 + M/60 + S/3600)/24\,. \end{aligned} \tag{4.18}$$

As an example, apply (4.16) and (4.18) to find the JD for the gregorian date 12:00 noon GMT, 1st January 2000 (sometimes referred to as 1.5 January); you'll find the result is exactly 2,451,545. You can verify that number by counting the days from first principles, being careful to remember that the year after 1 BC is AD 1: so 4713 BC is AD $-4712$, a leap year.

This well-defined time variable $t$ can now be used with the results of the previous sections to compute the planet's motion. An *epoch* $t_0$ is specified; this is any moment that astronomers agree to work from, usually not far in the past. The position of the planet at $t_0$ has been measured by astronomers and is specified as part of the set of parameters that describe the planet's orbit. We ask for the planet's position at time $t$; the answer will depend wholly on $t - t_0$. An easy way to compute the time $t - t_0$ elapsed since the epoch is to convert $t_0$ and $t$ to julian days JD($t_0$) and JD($t$). The time difference $t - t_0$ is then just JD($t$) − JD($t_0$)

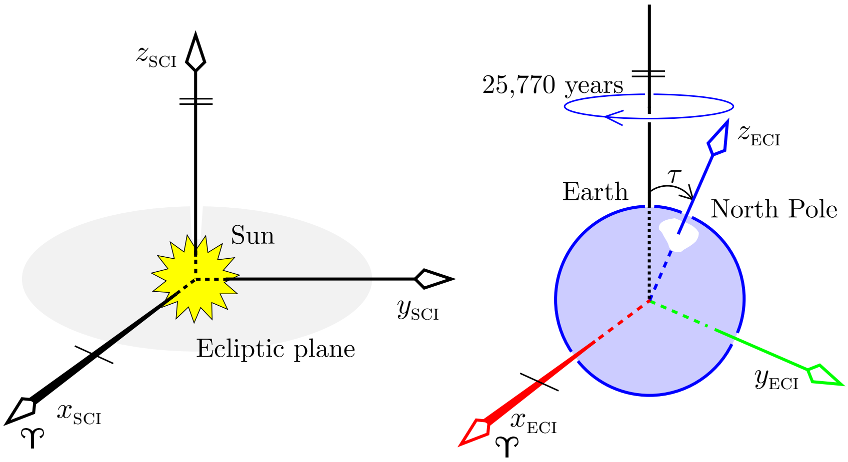



*Figure 3: Relative positioning of Sun-Centred Inertial coordinates SCI and Earth-Centred Inertial coordinates. The two directions marked with the short single lines are the same, as are the pair marked with the short double lines. Both $x_{SCI}$ and $x_{ECI}$ point to the First Point of Aries "♈", defined in Figure 4 and in the text. Earth tilts away from $z_{SCI}$ by its "polar tilt" $\tau \simeq 23.439°$. The small blue circle next to $z_{ECI}$ shows the direction of Earth's precession about the direction of $z_{SCI}$.*

"civilian days", where by a civilian day we mean normally exactly 24 hours; but if we require one-second accuracy, all leap seconds need to be taken into account too. We'll ignore leap seconds in this report.

Our constantly changing galaxy makes it impossible to predict arbitrarily far into the future and past, of course. You can think of the prediction of a planet's position at time $t$ as a zeroth- or sometimes first-order Taylor expansion about its measured position at $t_0$. We will assume that most parameters don't change with time, and will only use their first derivatives when necessary.

# 5 The Earth-Centred and Sun-Centred Inertial Frames

Describing planetary motion on the scale of the Solar System requires a choice of frames and coordinates [16]. Two useful inertial frames quantified by cartesian coordinates are shown in Figure 3. To a high precision their coordinate axes are unchanging, pointing to almost fixed points on the celestial sphere. The axes of the *Sun-Centred Inertial* (SCI) frame $x_{SCI}, y_{SCI}, z_{SCI}$ originate at the Sun. Earth's orbit defines the $x_{SCI}y_{SCI}$ plane (the *ecliptic plane*), and Earth orbits right handed about the $z_{SCI}$ axis.

The coordinate axes of the *Earth-Centred Inertial* (ECI) frame $x_{ECI}, y_{ECI}, z_{ECI}$ originate at Earth's centre. The $z_{ECI}$ axis is Earth's spin axis pointing out of our North Pole. The $x_{ECI}$ axis is parallel to the $x_{SCI}$ axis, and the $y_{ECI}$ axis completes the right-handed set. Note that Figure 3 deliberately shows Earth with no detail, to emphasise that the ECI axes bear no relation to Earth's surface features such as its countries.

The axes of the Sun-Centred Inertial frame are shown in more detail in Figure 4. The red–

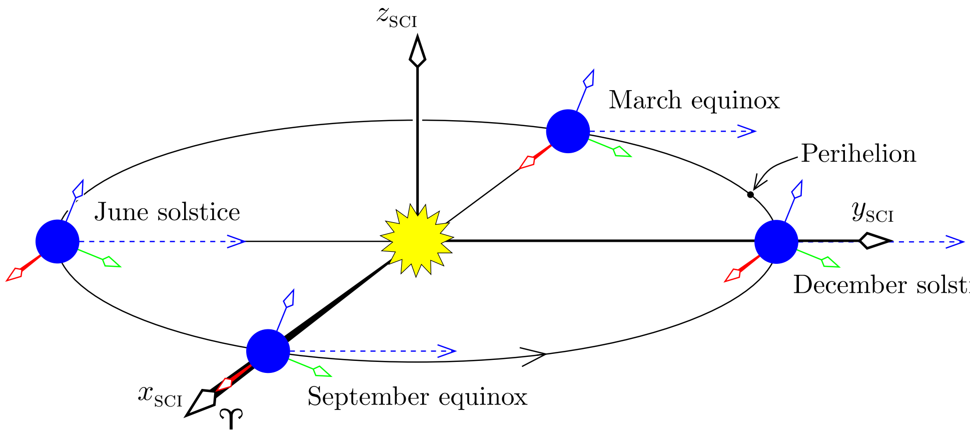



*Figure 4: Earth's orbit about the Sun, showing how all axes are defined. Earth carries its ECI axes almost without changing their orientation relative to the stars, apart from the slow change due to Earth's precession and nutation.*

green–blue axes in Figures 3 and 4 are the same set: ECI.

Earth's polar axis $z_{\text{ECI}}$ can be treated as a vector with a projection onto the ecliptic plane. This projection is drawn as the long dashed arrow in Figure 4; its length has been increased for clarity, but is otherwise irrelevant. Throughout the year this projection vector makes an angle with the Sun-to-Earth vector that varies from 0° to 360°. The angle is 0° in late December at a precise moment called the *December solstice*, when southern daylight is at its longest [17]. At the precise moment of the *March equinox*, again by definition, the angle has increased to 90°. At the *June solstice* it is 180°, and at the *September equinox* it is 270°. These key moments of time in Earth's orbit are defined by Earth's tilt, and so have nothing to do with the orbit's major and minor axes. Earth reaches perihelion in early January, when the Sun is 5 million km closer to us than it is in early July. The Sun's closeness in January makes its apparent area 7% bigger then, than in July.

These four instants in the year, two solstices and two equinoxes, are separated by 90° of true anomaly $\theta$; they are *not* separated by 90° of *mean* anomaly $M$, and so are not spaced exactly 3 months apart. But the eccentricity of Earth's orbit is low enough that they are spaced *almost* 3 months apart.

The $x_{\text{SCI}}$ (equivalently, $x_{\text{ECI}}$) axis is defined to lie along the Sun-to-Earth vector at the moment of the September equinox. This axis has a "vanishing point" in the sky which lies, by construction, on the *celestial equator*, being the projection of Earth's equatorial plane into the sky. This vanishing point is called the *First Point of Aries*, denoted ♈, the most important celestial measurement reference used by astronomers [18]. Why Aries? The vanishing point is almost fixed in the sky, but precession rotates Earth's spin axis $z_{\text{ECI}}$ left handed about $z_{\text{SCI}}$ once every 25,770 years, making the direction of $x_{\text{SCI}}$ (and $x_{\text{ECI}}$) change by 360° over this period [19]. The First Point thus moves slowly westward through the zodiacal constellations when viewed from Earth. In around 2000 BC the First Point moved from Taurus into Aries, and its name presumably derives from astrology. Precession has since carried it into Pisces, but it has kept its old name. Over the next few centuries the First Point will move into Aquarius, an event of astrological significance that spawned the catchy 1960s pop song "The [Dawning of the] Age of Aquarius".

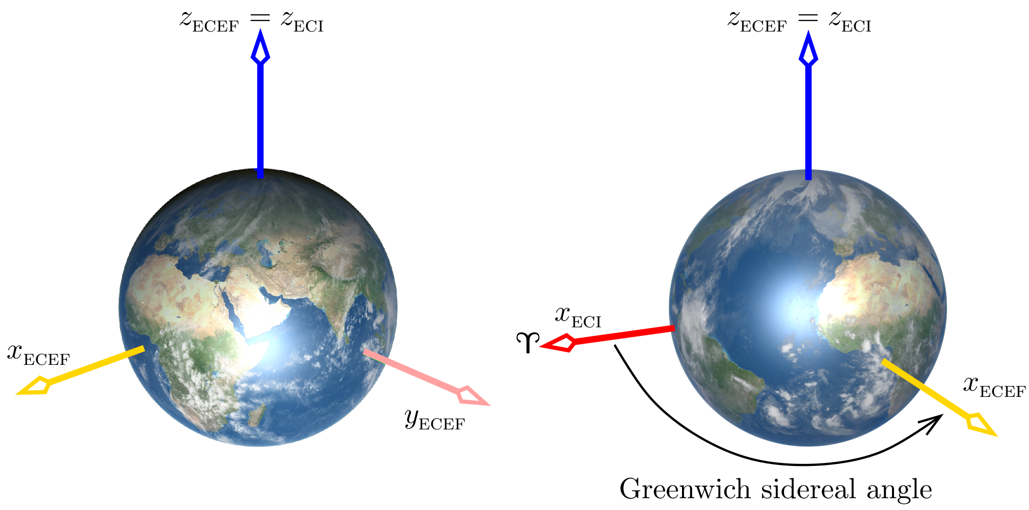



*Figure 5:* ***Left:*** *axes of the Earth-Centred Earth-Fixed frame are rigidly fixed to Earth's body in precisely the positions drawn. The* $x_{\text{ECEF}}$ *axis is located at the longitude of Greenwich.* ***Right:*** *at any moment, the amount by which Greenwich has turned past the First Point of Aries* ($x_{\text{ECI}}$) *is the "Greenwich sidereal angle".*

# 6 The ECEF and Sidereal Angle

The $x_{\text{SCI}}$ and $y_{\text{SCI}}$ axes slowly rotate as Earth precesses, but we can freeze the positions of those axes at some epoch and retain those positions for all future times of interest. When we run time forward or backward from that epoch to some requested moment of viewing the sky, these epoch SCI axes remain fixed by definition. Like these axes, the Earth-Centred Inertial axes $x_{\text{ECI}}, y_{\text{ECI}}, z_{\text{ECI}}$ remain almost fixed in space over time spans much less than 25,770 years. But we will certainly include the effect of precession by rotating Earth through the required angle within the "frozen" SCI frame.

Another frame of use—this one non-inertial—is the *Earth-Centred Earth-Fixed* (ECEF) frame, with cartesian coordinates shown in Figure 5. The ECEF's axes are fixed to Earth; the $x_{\text{ECEF}}$ axis points from Earth's centre through 0° latitude/0° longitude, the $z_{\text{ECEF}}$ axis coincides with the $z_{\text{ECI}}$ axis (Earth's spin axis), and $y_{\text{ECEF}}$ completes the set. Earth's spin within the Earth-Centred Inertial frame manifests as the ever-increasing angle between the "fixed" $x_{\text{ECI}}$ axis (pointing to ♈) and "rotating" $x_{\text{ECEF}}$ axis (at the longitude of Greenwich). This angle in Earth's equatorial plane is the *Greenwich sidereal angle*, and increases by 360° in the time it takes Earth to rotate once with respect to the distant stars, being one *sidereal day*, a period of 23 hours, 56 minutes, 4.09890 seconds [20].

Similar to the Greenwich sidereal angle, at any given moment the *local sidereal angle* of, say, Adelaide is the angle in Earth's equatorial plane from the $x_{\text{ECI}}$ axis to Adelaide's meridian (its great circle of longitude): this is just the Greenwich sidereal angle plus Adelaide's longitude. Knowing the current local sidereal angle equates to knowing our current orientation relative to the fixed stars, which allows us to determine where planets and stars will be seen in our sky right now.

To find the current local sidereal angle we need a start point; this is always the Greenwich sidereal angle at some epoch for which that angle has been measured to high preci-

sion. A standard epoch used here is "J2000.0" which uses the date mentioned in Section 4.1: 12:00 GMT 1st January 2000, at which moment the Greenwich sidereal angle—the angle in Earth's equatorial plane through which Earth's zero meridian had turned past the direction to the First Point of Aries—happened to be 280.46062°. (Remember that the Greenwich sidereal angle at time $t$ is a unique number independent of epoch; but to calculate it we must start somewhere: we must surely begin with an epoch for which we know the angle's value accurately.)

As an example, what is the sidereal angle of Adelaide at local (daylight savings) time 9:00 p.m. on 22nd March 2014? We simply add three angles modulo 360°. Begin at the J2000.0 epoch when the Greenwich angle was 280.46°, rotate Earth (and hence Greenwich) from the epoch to the current time through the second angle, and then add Adelaide's longitude of 138.60°. The time through which we must rotate Earth is the requested time minus the epoch. Work in GMT: convert Adelaide's time to 10:30 a.m. GMT 22nd March 2014, then use (4.16) and (4.18) to find that this is JD 2,456,738.9375. The time elapsed since the J2000.0 epoch is this JD minus 2,451,545, or 5193.9375 days. The angle turned through by Earth relative to the First Point (modulo 360°) since the epoch is then its hourly rate times the number of hours elapsed:

$$\frac{360^\circ}{23\,\mathrm{h}\;56\,\mathrm{m}\;4.09890\,\mathrm{s}} \times 5193.9375 \times 24\,\mathrm{h} \simeq 56.71^\circ. \tag{6.1}$$

So the sidereal angle of Adelaide at local time 9:00 p.m. on 22nd March 2014 is

$$\underbrace{\underbrace{280.46^\circ}_{\text{angle at epoch}} + \underbrace{56.71^\circ}_{\text{Earth turned}}}_{\text{Greenwich sidereal angle}} + \underbrace{138.60^\circ}_{\text{Adelaide's longitude}} = 115.77^\circ \quad (\text{modulo } 360^\circ \text{ of course}). \tag{6.2}$$

At this time the First Point of Aries lies 115.77° west of the Adelaide meridian, measured along the celestial equator. Similarly, at any time and place on Earth the local sidereal angle is really just a measure of where we see the stars to be. As the stars wheel about the celestial poles, any one of them can be visualised as being at the end of a giant clock hand that turns through 360° in the above 23-56-4.09890 hours. This is probably what has prompted astronomers to call the sidereal angle the sidereal *time*. With this name, the angle is usually specified in hours/minutes/seconds, where 24 such angular hours are defined to be exactly 360°; that is, 24 sidereal hours of angle are turned through by Earth relative to the inertial stars in the above 23-56-4.09890 civil time hours. My view is that the name "sidereal time" is an unfortunate choice for something that is simply an angle and is always used geometrically; the fact that the spinning Earth acts like a giant clock does not call for a new unit of time. Also, the use of a time unit to measure this angle is unfortunate because a sidereal hour has no useful relation to a civil hour of 3600 seconds, and even when "sidereal time" is quoted in hours/minutes/seconds, it still denotes an angle, not a real time. In this report, "time" denotes real time and "angle" denotes angle, and all hours are the only sort that I maintain should exist: civil time hours, the sort that clocks and wrist watches measure [21].

On a related note, if you prefer using degrees instead of radians, do write your angles as one decimal number of degrees. Just like pounds–shillings–pence, expressing angles in base 60 as degrees–minutes–seconds is an old but clunky practice, one that only obfuscates calculations.

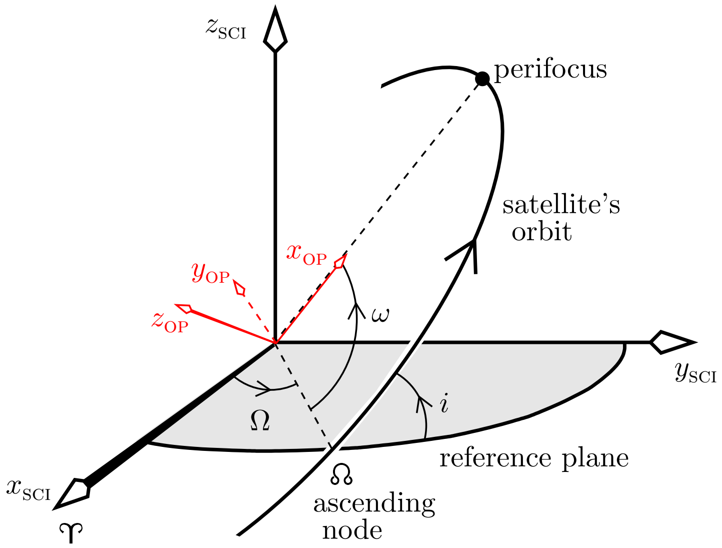



*Figure 6: The elements describing an orbit's orientation relative to a reference plane such as the ecliptic*

# 7 Orbital Elements

Having located a planet within its orbit at some requested time, we must now place the orbit in its correct orientation within the Solar System. The orbit is typically described by three angles shown in Figure 6 that refer it to some convenient plane such as the ecliptic. The point where the planet crosses the ecliptic plane in the direction of increasing $z_{\text{SCI}}$ is called its *ascending node*, denoted ☊. Two angles, the *orbital inclination* $i$ and the *longitude of the ascending node* $\Omega$ locate the orbit plane relative to the ecliptic, and a third angle, the *argument of the perifocus* $\omega$, locates the perifocus. A fourth angle (one of the "anomalies", usually $M$) specifies the planet's position in its orbit at some epoch $t_0$, which need not be the same as the epoch used above for calculating sidereal angle. Together with the semi-major axis length $a$ and eccentricity $e$, these angles $i, \Omega, \omega$, and $M_0 = M(t_0)$ comprise a set of six parameters that fully describe an orbit and where the planet is located in it at the epoch $t_0$. They are determined from observations by astronomers, and we'll take them as given. Their first derivatives are also available, so that whereas we'll obtain enough precision by taking e.g. $\Omega$ to be constant for each planet, we could also use $\Omega(t) \simeq \Omega(t_0) + \dot{\Omega}(t_0)(t - t_0)$ for higher precision. Remember that the longitude $\Omega$ for planets orbiting the Sun is an angle in the *ecliptic* plane $x_{\text{SCI}}\, y_{\text{SCI}}$, measured east of the First Point of Aries—as opposed to the everyday terrestrial longitude of world cities, which is measured east of the Greenwich meridian in Earth's *equatorial* plane $x_{\text{ECEF}}\, y_{\text{ECEF}}$.

Note the three words that all denote an angle here: longitude, argument, anomaly. Perhaps these words date back to those maritime practices of old referred to earlier. And perhaps it's those same die-hard practices that have caused $\omega$ and $M_0$ to be sometimes scrambled mathematically, and pointlessly. A "longitude of perifocus" is defined as $\Omega + \omega$, which makes no mathematical sense because $\Omega$ and $\omega$ lie in different planes. (It is meaningless to add angles

in different planes because the sum retains no information on the contribution from each plane, and so can't distinguish between different geometrical situations. That is, an orbit described by $\Omega = 50°$ and $\omega = 20°$ is quite different to an orbit described by $\Omega = 40°$ and $\omega = 30°$.) Also a "mean longitude" is defined as $\Omega + \omega + M_0$, a doubly meaningless quantity because it adds a mean anomaly, which is not even an angle that relates directly to the planet's position! Despite their names, neither of these two bizarre sums are angles, and they are *only* ever used as a kind of secret code requiring decoding with a "key" $\Omega$, which is always given and is used to extract $\omega$ along with $M_0$:

$$\begin{aligned} \omega &= \text{long. of perifocus} - \Omega\,, \\ M_0 &= \text{mean longitude} - \text{long. of perifocus}. \end{aligned} \tag{7.1}$$

I won't dignify the longitude of perifocus and mean longitude with mathematical symbols, and I wish celestial navigators would stop using these mis-defined and mis-named quantities. These two parameters have no use other than to require decrypting to produce the actual parameters needed, $\omega$ and $M_0$ via (7.1). Aside from their lack of mathematical meaning, they simplify nothing, neither in computation nor in understanding the subject. The longitude of perifocus and mean longitude are sometimes argued for on a supposed non-definition of $\omega$ for circular orbits (which have no perifocus) or $\Omega$ for equatorial orbits (which have no ascending node). But the line of mathematical reasoning that constructs an orbit does so by starting with a conic section and referring it to $x_{\text{OP}}, y_{\text{OP}}, z_{\text{OP}}$ axes—which can certainly be done even for a circular orbit with its absence of perifocus. These axes then define $\omega$. Also, whilst an equatorial orbit has no ascending node, it can still be given a value of $\Omega$, in fact *any* value, since $\Omega$ serves only to describe how the orbit is orientated. In such a case the sum $\Omega + \omega$ *is* well defined because $\Omega$ and $\omega$ are in the same plane, and increasing the choice of $\Omega$ by, say, $1°$ must be offset by decreasing the choice of $\omega$ by $1°$. So only in that case of an equatorial orbit does $\Omega + \omega$ have any meaning—but it is not needed there.

Whether the longitude of perifocus and mean longitude have ever prevented a ship's mutiny might never be known. But those days are long gone, and I think the two quantities should now be relegated to the history of orbital mechanics [22].

# 8 Relating Coordinate Systems

We will need various cartesian coordinate systems to locate a planet in our night sky. Although the following calculations can easily be combined in a way that enables some of these coordinates to be omitted, it's advantageous to employ more coordinate systems than are strictly necessary, as a way of separating the calculations into manageable steps.

First, orbit-plane coordinates centred on the Sun describe the planet's motion most simply: the planet moves right handed about $z_{\text{OP}}$ in the $x_{\text{OP}}\, y_{\text{OP}}$ plane with polar coordinates $r, \theta$. The perifocus lies on the positive $x_{\text{OP}}$ axis. Take careful note that orbit-plane coordinates are planet dependent, so we write e.g. $x_{\text{OPJ}}$ for Jupiter and $x_{\text{OPE}}$ for Earth.

Next, Sun-Centred Inertial coordinates SCI describe the unchanging "global" Solar System frame, also centred on the Sun, discussed in Section 5. *Local* coordinates such as "east–north–up" (ENU) are centered on the Earth observer, and are easily converted to local bearing and elevation. We could use just these coordinate sets, but will also employ ECI and ECEF coordinates as intermediate steps.

Here is a very useful way to keep track and make sense of the various vectors and their coordinates within these systems. First, remember that a position is always specified relative to some point. To sight Jupiter $J$ from Adelaide $A$, we require the position vector $\boldsymbol{r}_{JA}$ of Jupiter relative to Adelaide. The orbital parameters for Jupiter will give us the position of Jupiter $\boldsymbol{r}_{JS}$ relative to the Sun $S$, and the position $\boldsymbol{r}_{ES}$ of Earth's *centre* $E$ relative to the Sun. Although it isn't necessary for a calculation involving Jupiter, we will carefully distinguish Earth's centre from the location of Adelaide, so will require the position of Adelaide $\boldsymbol{r}_{AE}$ relative to Earth's centre. This last vector is really quite negligible, but we include it to show how the various pieces of the jigsaw fit together. (For sighting Jupiter from Adelaide, the angular error caused by omitting $\boldsymbol{r}_{AE}$ is quickly estimated as Earth's radius divided by Jupiter's distance from the Sun, or approximately 6400 km/(780 million km), or about 0.0005°.)

The above vectors are related via

$$\begin{aligned} \boldsymbol{r}_{JA} &= \boldsymbol{r}_{JS} + \boldsymbol{r}_{SE} + \boldsymbol{r}_{EA} \\ &= \boldsymbol{r}_{JS} - \boldsymbol{r}_{ES} - \boldsymbol{r}_{AE}\,. \end{aligned} \tag{8.1}$$

Notice that no numbers are present in (8.1): it holds independently of any coordinate choice. $\boldsymbol{r}_{JA}$ is a *proper vector*, an arrow, but it has coordinates in any coordinate system $C$ that we choose. These numbers form a 3-element *coordinate vector* $[\boldsymbol{r}_{JE}]_C$, which we write as a column of numbers, useful later for multiplying it by a matrix. This notation was introduced in Section 2.

Our central task is to relate the coordinates of $\boldsymbol{r}_{JE}$ in any two different cartesian coordinate systems $C$ and $C'$. That is, given the $C$-coordinates $[\boldsymbol{v}]_C$ of some vector $\boldsymbol{v}$, what are its $C'$-coordinates $[\boldsymbol{v}]_{C'}$? The following analysis is a standard technique of linear algebra, but is included here for completeness. The use of just two dimensions here saves space, but the same results apply in three dimensions.

Write the unit basis vectors of $C$ as $\boldsymbol{u}_x, \boldsymbol{u}_y$ and the unit basis vectors of $C'$ as $\boldsymbol{u}_{x'}, \boldsymbol{u}_{y'}$. The following procedure demands each set of basis vectors to be *orthonormal*, meaning its vectors have unit length and are mutually orthogonal. Starting with

$$\boldsymbol{v} = v_x\boldsymbol{u}_x + v_y\boldsymbol{u}_y = v_{x'}\boldsymbol{u}_{x'} + v_{y'}\boldsymbol{u}_{y'}\,, \tag{8.2}$$

the relevant coordinate vectors of $\boldsymbol{v}$ are

$$[\boldsymbol{v}]_C = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}\cdot\boldsymbol{u}_x \\ \boldsymbol{v}\cdot\boldsymbol{u}_y \end{bmatrix}, \qquad [\boldsymbol{v}]_{C'} = \begin{bmatrix} v_{x'} \\ v_{y'} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}\cdot\boldsymbol{u}_{x'} \\ \boldsymbol{v}\cdot\boldsymbol{u}_{y'} \end{bmatrix}. \tag{8.3}$$

Now consider

$$\begin{aligned} [\boldsymbol{v}]_{C'} = \begin{bmatrix} \boldsymbol{v}\cdot\boldsymbol{u}_{x'} \\ \boldsymbol{v}\cdot\boldsymbol{u}_{y'} \end{bmatrix} &= \begin{bmatrix} (v_x\boldsymbol{u}_x + v_y\boldsymbol{u}_y)\cdot\boldsymbol{u}_{x'} \\ (v_x\boldsymbol{u}_x + v_y\boldsymbol{u}_y)\cdot\boldsymbol{u}_{y'} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{u}_x\cdot\boldsymbol{u}_{x'} & \boldsymbol{u}_y\cdot\boldsymbol{u}_{x'} \\ \boldsymbol{u}_x\cdot\boldsymbol{u}_{y'} & \boldsymbol{u}_y\cdot\boldsymbol{u}_{y'} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} \equiv \mu^C_{C'}\,[\boldsymbol{v}]_C\,. \end{aligned} \tag{8.4}$$

That is, the matrix $\mu^C_{C'}$ transforms coordinates from $C$ to $C'$:

$$[\boldsymbol{v}]_{C'} = \mu^C_{C'}\,[\boldsymbol{v}]_C\,, \tag{8.5}$$

where (8.4) shows that the *columns* of $\mu^C_{C'}$ are the basis vectors of $C$ expressed in $C'$ coordinates:

$$\mu^C_{C'} \equiv \Big[ [\boldsymbol{u}_x]_{C'} \;\; [\boldsymbol{u}_y]_{C'} \Big]. \tag{8.6}$$

We will use the idea of (8.6) to construct the necessary $\mu$ matrices ahead [23]. Equivalently, (8.4) shows that the *rows* of $\mu^C_{C'}$ are the basis vectors of $C'$ expressed in $C$ coordinates. From this, it's not hard to see that $\mu^{C'}_C$ is both the transpose and the inverse of $\mu^C_{C'}$:

$$\mu^{C'}_C = \left(\mu^C_{C'}\right)^t = \left(\mu^C_{C'}\right)^{-1}. \tag{8.7}$$

So a $\mu$ matrix is inverted by transposing it. These matrices allow a switch between any coordinates that we prefer to use. They can be "chained" together; for example, with three coordinate systems:

$$[\boldsymbol{v}]_A = \mu^B_A\,[\boldsymbol{v}]_B = \mu^B_A\,\mu^C_B\,[\boldsymbol{v}]_C \equiv \mu^C_A\,[\boldsymbol{v}]_C\,, \tag{8.8}$$

showing that $\mu^C_A = \mu^B_A\,\mu^C_B$. The extension to more coordinate systems is immediate, so that for coordinate systems $A, B, \ldots, Y, Z$,

$$\mu^Z_A = \mu^B_A\,\mu^C_B\,\mu^D_C\,\ldots\,\mu^Z_Y\,. \tag{8.9}$$

Calculating a $\mu$ matrix presents a slight complication: the approach that might be considered as intuitive requires more calculation than is needed by a particular less intuitive approach. Fortunately we can always convert each of these approaches to the other, depending on whether we seek more intuition or more mathematical simplicity (which equates to computational speed here). As an example, use (8.6) as a guide to calculate $\mu^{\text{OP}}_{\text{SCI}}$, which converts orbit-plane coordinates to Sun-Centred Inertial coordinates:

$$\mu^{\text{OP}}_{\text{SCI}} = \Big[ [\boldsymbol{u}_{x_{\text{OP}}}]_{\text{SCI}} \;\; [\boldsymbol{u}_{y_{\text{OP}}}]_{\text{SCI}} \;\; [\boldsymbol{u}_{z_{\text{OP}}}]_{\text{SCI}} \Big]. \tag{8.10}$$

Before we describe the rotations, it helps the visualisation to realise that a proper vector connecting two well-defined points is not tied to the origin of any set of coordinates; you can translate that vector anywhere at all if that helps you to visualise what we are doing. So to picture a rotation through some angle, imagine the vectors as moving over the surface of a sphere through that angle, all the while being rigidly embedded in the local tangent plane to the sphere's surface; this is richer than simply imagining them to be rotating with their tails anchored to some point on the rotation axis. When the rotations involve latitude/longitude, you can also picture the vectors moving over Earth's surface along small or great circles, while being rigidly embedded in the surface's tangent plane; although Earth isn't quite spherical, the commonly used definition of latitude, *geodetic*, ensures that moving a vector in this way along a meridian between two latitudes will rotate it correctly through the angular difference of the latitudes; for example, this is why equation (8.26) ahead is exact.

Referring to Figure 6, imagine taking copies of the SCI basis vectors $\boldsymbol{u}_{x_{\text{SCI}}}, \boldsymbol{u}_{y_{\text{SCI}}}, \boldsymbol{u}_{z_{\text{SCI}}}$ and rotating these copies through the angles $i, \Omega, \omega$ to become the orbit-plane set $\boldsymbol{u}_{x_{\text{OP}}}, \boldsymbol{u}_{y_{\text{OP}}}, \boldsymbol{u}_{z_{\text{OP}}}$. (Remember that all rotations are right handed unless otherwise stated.) Take care to get the rotation order right, because two rotations cannot generally be swapped. Picture the sequence carefully [24]: first rotate each copy of $\boldsymbol{u}_{x_{\text{SCI}}}, \boldsymbol{u}_{y_{\text{SCI}}}, \boldsymbol{u}_{z_{\text{SCI}}}$ around $z_{\text{SCI}}$ by $\Omega$, then rotate the results around the *rotated version of* $\boldsymbol{u}_{x_{\text{SCI}}}$ by $i$, and finally rotate the results around the *doubly rotated version of* $\boldsymbol{u}_{z_{\text{SCI}}}$ by $\omega$, producing $\boldsymbol{u}_{x_{\text{OP}}}, \boldsymbol{u}_{y_{\text{OP}}}, \boldsymbol{u}_{z_{\text{OP}}}$. Write this procedure as

$$\begin{aligned}\{\boldsymbol{u}_{x_{\text{SCI}}}, \boldsymbol{u}_{y_{\text{SCI}}}, \boldsymbol{u}_{z_{\text{SCI}}}\} &\to R^{\Omega}_{z_{\text{SCI}}} \to R^{i}_{\langle x_{\text{SCI}}\rangle} \to R^{\omega}_{\langle z_{\text{SCI}}\rangle} \\ &\to \{\boldsymbol{u}_{x_{\text{OP}}}, \boldsymbol{u}_{y_{\text{OP}}}, \boldsymbol{u}_{z_{\text{OP}}}\}\,,\end{aligned} \tag{8.11}$$

where $R^{\Omega}_{z_{\text{SCI}}}$ means "rotate about $z_{\text{SCI}}$ (or $\boldsymbol{u}_{z_{\text{SCI}}}$) by $\Omega$", and $\langle x_{\text{SCI}}\rangle$ denotes the latest-rotated copy of the $x_{\text{SCI}}$ basis vector.

This procedure is easy to visualise, but requires the carried-along axes to be calculated because we are to rotate around them. That's not difficult to do, but much easier and much faster computationally is to realise that the rotations can be done around the *original* SCI axes/basis vectors only: first rotate the copies of $\boldsymbol{u}_{x_{\text{SCI}}}, \boldsymbol{u}_{y_{\text{SCI}}}, \boldsymbol{u}_{z_{\text{SCI}}}$ around $z_{\text{SCI}}$ by $\omega$, then rotate the results around $x_{\text{SCI}}$ by $i$, and finally rotate the results around $z_{\text{SCI}}$ by $\Omega$:

$$\begin{aligned}\{\boldsymbol{u}_{x_{\text{SCI}}}, \boldsymbol{u}_{y_{\text{SCI}}}, \boldsymbol{u}_{z_{\text{SCI}}}\} &\to R^{\omega}_{z_{\text{SCI}}} \to R^{i}_{x_{\text{SCI}}} \to R^{\Omega}_{z_{\text{SCI}}} \\ &\to \{\boldsymbol{u}_{x_{\text{OP}}}, \boldsymbol{u}_{y_{\text{OP}}}, \boldsymbol{u}_{z_{\text{OP}}}\}\,. \end{aligned} \tag{8.12}$$

This last description is perhaps not so intuitively obvious as the first one in (8.11). Compare (8.11) and (8.12): equation (8.12) reverses the order of the angles rotated through in (8.11) while changing the sense of its rotation axes from "copies carried along" to "fixed originals". It's quite easy to show that this reversal can *always* be done for any number of rotations, even if the rotation axes are not mutually perpendicular. This apparently nameless theorem simplifies many calculations in orientational analysis but, surprisingly, is not well known. Reference [25] discusses it in detail.

Rotating a proper vector about an $x$ axis though angle $\theta$ is accomplished by multiplying its coordinate vector by the *Euler matrix* $E^{\theta}_1$, and similarly for rotations about the $y$ and $z$ axes [26]:

$$E^{\theta}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad E^{\theta}_2 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad E^{\theta}_3 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.13}$$

Recall that we are calculating $\mu^{\text{OP}}_{\text{SCI}}$ as an example of the method of constructing these $\mu$ matrices. Refer now to (8.10) and realise that each column of $\mu^{\text{OP}}_{\text{SCI}}$ is one of the OP basis vectors expressed in SCI coordinates. So follow the procedure of (8.12) to construct the OP basis vectors, all the while working in SCI coordinates. Three rotations are required, carried out by three matrix multiplications (note the correct order!):

$$\begin{aligned} [\boldsymbol{u}_{x_{\text{OP}}}]_{\text{SCI}} = E^{\Omega}_3\, E^{i}_1\, E^{\omega}_3 \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad [\boldsymbol{u}_{y_{\text{OP}}}]_{\text{SCI}} = E^{\Omega}_3\, E^{i}_1\, E^{\omega}_3 \begin{bmatrix}0\\1\\0\end{bmatrix}, \\ [\boldsymbol{u}_{z_{\text{OP}}}]_{\text{SCI}} = E^{\Omega}_3\, E^{i}_1\, E^{\omega}_3 \begin{bmatrix}0\\0\\1\end{bmatrix}, \end{aligned} \tag{8.14}$$

from which matrix block multiplication makes it apparent that

$$\begin{aligned}\mu^{\text{OP}}_{\text{SCI}} &= \Big[ [\boldsymbol{u}_{x_{\text{OP}}}]_{\text{SCI}} \quad [\boldsymbol{u}_{y_{\text{OP}}}]_{\text{SCI}} \quad [\boldsymbol{u}_{z_{\text{OP}}}]_{\text{SCI}} \Big] \\ &= E^{\Omega}_3\, E^{i}_1\, E^{\omega}_3 \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = E^{\Omega}_3\, E^{i}_1\, E^{\omega}_3\,. \end{aligned} \tag{8.15}$$

## 8.1 Combining the Various Coordinate Systems

The bearing and elevation of Jupiter in Adelaide's sky [27] can easily be derived from the ENU coordinates of the position vector of Jupiter relative to Adelaide, so we require to

calculate $[\boldsymbol{r}_{JA}]_{\text{ENU}}$ by using (8.1):

$$\begin{aligned}[\boldsymbol{r}_{JA}]_{\text{ENU}} &= [\boldsymbol{r}_{JS} - \boldsymbol{r}_{ES} - \boldsymbol{r}_{AE}]_{\text{ENU}} \\ &= \mu_{\text{ENU}}^{\text{OPJ}}\,[\boldsymbol{r}_{JS}]_{\text{OPJ}} - \mu_{\text{ENU}}^{\text{OPE}}\,[\boldsymbol{r}_{ES}]_{\text{OPE}} - \mu_{\text{ENU}}^{\text{ECEF}}\,[\boldsymbol{r}_{AE}]_{\text{ECEF}}\,, \end{aligned} \tag{8.16}$$

where OPJ denotes the orbit-plane coordinate system of Jupiter and OPE is likewise for Earth. Equation (4.2) gives $[\boldsymbol{r}_{JS}]_{\text{OPJ}}$ and $[\boldsymbol{r}_{ES}]_{\text{OPE}}$, where each of these are the transpose of $(x_{\text{OP}}, y_{\text{OP}}, 0)$ in that equation, with $x_{\text{OP}}, y_{\text{OP}}$ calculated in each instance from the orbital parameters of Jupiter and Earth respectively. The latitude/longitude of Adelaide suffice to give $[\boldsymbol{r}_{AE}]_{\text{ECEF}}$.

Here are the details of calculating the three coordinate vectors in (8.16). Jupiter's orbital elements are $a, e, i, \Omega, \omega, M_0$ at some epoch $t_0$. Calculate the semi-minor axis length $b$ from $a$ and $e$ using the expression in the text just after (3.2). Calculate Jupiter's period $T$ from (3.3). Convert the epoch time $t_0$ to a JD, and do likewise for the requested time $t$ to find $t - t_0$. Then evolve the mean anomaly using Kepler's equation (4.6), written as

$$M = M_0 + 2\pi(t - t_0)/T\,, \tag{8.17}$$

remembering to replace the $2\pi$ by 360° if you are working in degrees. Produce $E$ by solving $E - e\sin E = M$ (being careful to use radians here). Now invoke (4.2) to write

$$[\boldsymbol{r}_{JS}]_{\text{OPJ}} = \begin{bmatrix} x_{\text{OPJ}} \\ y_{\text{OPJ}} \\ 0 \end{bmatrix} = \begin{bmatrix} a(\cos E - e) \\ b\sin E \\ 0 \end{bmatrix}. \tag{8.18}$$

The same calculation using Earth's orbital elements produces $[\boldsymbol{r}_{ES}]_{\text{OPE}}$. A reasonably accurate version of the position vector of a city (say, Adelaide) relative to Earth's centre in ECEF coordinates is given by

$$[\boldsymbol{r}_{AE}]_{\text{ECEF}} = R \begin{bmatrix} \cos\lambda\cos\phi \\ \cos\lambda\sin\phi \\ \sin\lambda \end{bmatrix}, \tag{8.19}$$

where that city has latitude $\lambda$ and longitude $\phi$, and Earth is assumed spherical with radius $R$. This is sufficient for producing Jupiter's sight direction from Adelaide, but a more accurate expression is needed for working with satellites that orbit Earth. This more accurate version models Earth as an oblate spheroid using the WGS-84 set of Earth dimensions. Write Earth's spheroidal radii as

$$\text{equatorial: } a = 6{,}378{,}137 \text{ m}\,, \quad \text{polar: } b = 6{,}356{,}752.3142 \text{ m}. \tag{8.20}$$

Location $A$ lies at latitude $\lambda$, longitude $\phi$, and height $h$ above the WGS-84 spheroid (that is, $h$ is approximate height above mean sea level). Write

$$k \equiv \sqrt{a^2\cos^2\lambda + b^2\sin^2\lambda}\,. \tag{8.21}$$

Then $A$ has a position relative to Earth's centre of

$$[\boldsymbol{r}_{AE}]_{\text{ECEF}} = \begin{bmatrix} (a^2/k + h)\cos\lambda\cos\phi \\ (a^2/k + h)\cos\lambda\sin\phi \\ (b^2/k + h)\sin\lambda \end{bmatrix}. \tag{8.22}$$

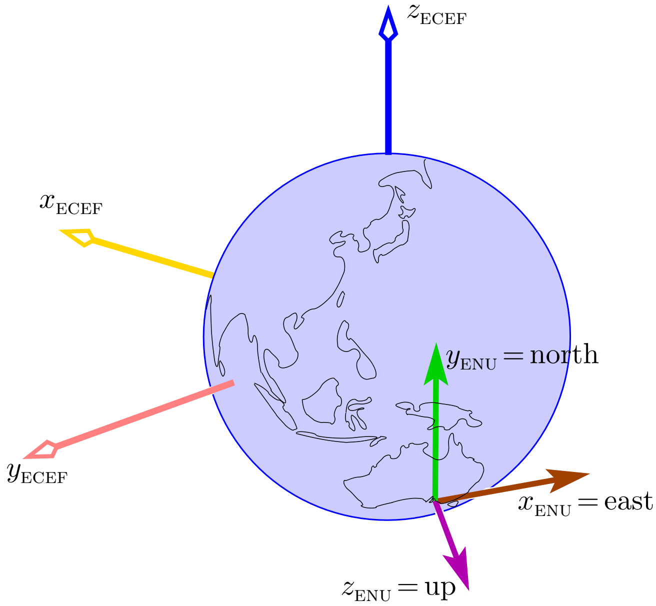



*Figure 7: Relative orientations of the* ECEF *and* ENU *axes. The colours of the* ECEF *axes here match those in Figure 5.*

The transformation matrices in (8.16), $\mu^{\text{OPJ}}_{\text{ENU}}, \mu^{\text{OPE}}_{\text{ENU}}$, and $\mu^{\text{ECEF}}_{\text{ENU}}$, can each be calculated with one sequence of rotations, but for clarity and to avoid duplication of effort, we will break the calculation of $\mu^{\text{OPJ}}_{\text{ENU}}$ and $\mu^{\text{OPE}}_{\text{ENU}}$ into several steps. These two matrices are just the two necessary instances of the generic matrix $\mu^{\text{OPX}}_{\text{ENU}}$ that converts orbit-plane coordinates for planet X (meaning Jupiter or Earth) to ENU. Write

$$\mu^{\text{OPX}}_{\text{ENU}} = \mu^{\text{ECEF}}_{\text{ENU}}\, \mu^{\text{ECI}}_{\text{ECEF}}\, \mu^{\text{SCI}}_{\text{ECI}}\, \mu^{\text{OPX}}_{\text{SCI}}\,, \tag{8.23}$$

and calculate each of the transformation matrices on the right-hand side of (8.23). Taken as a group, these calculations convert the coordinate systems step-by-step in the following chain:

Figure 3 (over SCI ⟶ ECI) — Figure 7 (over ECEF ⟶ ENU)

$$\text{OP} \longrightarrow \text{SCI} \longrightarrow \text{ECI} \longrightarrow \text{ECEF} \longrightarrow \text{ENU}$$

Figure 6 (under OP ⟶ SCI) — Figure 5 (under ECI ⟶ ECEF)

**Calculate $\mu^{\text{ECEF}}_{\text{ENU}}$:**

$$\mu^{\text{ECEF}}_{\text{ENU}} = \Big[ [\boldsymbol{u}_{x_{\text{ECEF}}}]_{\text{ENU}} \quad [\boldsymbol{u}_{y_{\text{ECEF}}}]_{\text{ENU}} \quad [\boldsymbol{u}_{z_{\text{ECEF}}}]_{\text{ENU}} \Big]. \tag{8.24}$$

We'll find it easier to calculate the transpose of $\mu^{\text{ECEF}}_{\text{ENU}}$, which is $\mu^{\text{ENU}}_{\text{ECEF}}$:

$$\mu^{\text{ENU}}_{\text{ECEF}} = \Big[ [\boldsymbol{u}_{x_{\text{ENU}}}]_{\text{ECEF}} \quad [\boldsymbol{u}_{y_{\text{ENU}}}]_{\text{ECEF}} \quad [\boldsymbol{u}_{z_{\text{ENU}}}]_{\text{ECEF}} \Big]. \tag{8.25}$$

Relative orientations of the ECEF and ENU axes are shown in Figure 7. The ENU coordinates have their origin at Adelaide, at latitude $\lambda$, longitude $\phi$. The sequence of rotations that takes

copies of the ENU basis vectors and rotates them to becomes ECEF basis vectors is (noting carefully the initial order)

$$\{\boldsymbol{u}_{y_{\text{ECEF}}}, \boldsymbol{u}_{z_{\text{ECEF}}}, \boldsymbol{u}_{x_{\text{ECEF}}}\} \to R^{-\lambda}_{y_{\text{ECEF}}} \to R^{\phi}_{z_{\text{ECEF}}} \to \{\boldsymbol{u}_{x_{\text{ENU}}}, \boldsymbol{u}_{y_{\text{ENU}}}, \boldsymbol{u}_{z_{\text{ENU}}}\}\,. \tag{8.26}$$

(Or imagine rotating with $R^{\phi}_{z_{\text{ECEF}}}$ then $R^{-\lambda}_{\langle y_{\text{ECEF}}\rangle}$, then apply the pseudo-reversing theorem.) Recall the earlier comment that even though Earth isn't quite spherical, the fact that we are using *geodetic* latitude ensures that (8.26) is exact.

Working in ECEF coordinates, (8.25) becomes

$$\mu^{\text{ENU}}_{\text{ECEF}} = R^{\phi}_{z_{\text{ECEF}}}\, R^{-\lambda}_{y_{\text{ECEF}}} \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix} = E^{\phi}_3\, E^{-\lambda}_2 \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}, \tag{8.27}$$

and it follows that the sought-after transformation matrix is

$$\mu^{\text{ECEF}}_{\text{ENU}} = \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix} E^{\lambda}_2\, E^{-\phi}_3. \tag{8.28}$$

**Calculate $\mu^{\text{ECI}}_{\text{ECEF}}$:** Again work with the transpose. The ECEF axes are rotated from the ECI axes by the Greenwich sidereal angle $\gamma$ around the $z$ axis shared by ECEF and ECI coordinates, so

$$\mu^{\text{ECEF}}_{\text{ECI}} = \Big[ [\boldsymbol{u}_{x_{\text{ECEF}}}]_{\text{ECI}} \quad [\boldsymbol{u}_{y_{\text{ECEF}}}]_{\text{ECI}} \quad [\boldsymbol{u}_{z_{\text{ECEF}}}]_{\text{ECI}} \Big] = R^{\gamma}_{z_{\text{ECI}}} \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} = E^{\gamma}_3\,, \tag{8.29}$$

and the required transformation matrix is

$$\mu^{\text{ECI}}_{\text{ECEF}} = E^{-\gamma}_3. \tag{8.30}$$

**Calculate $\mu^{\text{SCI}}_{\text{ECI}}$:** Again, calculate $\mu^{\text{ECI}}_{\text{SCI}}$ first:

$$\mu^{\text{ECI}}_{\text{SCI}} = \Big[ [\boldsymbol{u}_{x_{\text{ECI}}}]_{\text{SCI}} \quad [\boldsymbol{u}_{y_{\text{ECI}}}]_{\text{SCI}} \quad [\boldsymbol{u}_{z_{\text{ECI}}}]_{\text{SCI}} \Big]\,. \tag{8.31}$$

Referring to Figure 3, tilt Earth by rotating copies of the SCI basis vectors left-handed around $x_{\text{SCI}}$ by Earth's tilt $\tau$ (i.e. right-handed by $-\tau$), then rotate the resulting vectors left-handed around the original $z_{\text{SCI}}$ through the precession angle $p = 360°(t - t_0)/25{,}770$ years (i.e. right handed by $-p$). The rotations are

$$\{\boldsymbol{u}_{x_{\text{SCI}}}, \boldsymbol{u}_{y_{\text{SCI}}}, \boldsymbol{u}_{z_{\text{SCI}}}\} \to R^{-\tau}_{x_{\text{SCI}}} \to R^{-p}_{z_{\text{SCI}}} \to \{\boldsymbol{u}_{x_{\text{ECI}}}, \boldsymbol{u}_{y_{\text{ECI}}}, \boldsymbol{u}_{z_{\text{ECI}}}\}\,, \tag{8.32}$$

(or consider precession followed by tilt, and apply the pseudo-reversing theorem), so that

$$\mu^{\text{ECI}}_{\text{SCI}} = R^{-p}_{z_{\text{SCI}}}\, R^{-\tau}_{x_{\text{SCI}}} \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} = E^{-p}_3\, E^{-\tau}_1\,. \tag{8.33}$$

Now transpose to produce the required matrix:

$$\mu^{\text{SCI}}_{\text{ECI}} = E^{\tau}_1\, E^{p}_3\,. \tag{8.34}$$

**Calculate $\mu^{\text{OPX}}_{\text{SCI}}$:** This was obtained earlier in (8.15), where the $\Omega, i, \omega$ in that equation are those for planet X.

We can avoid some duplication of effort by writing (8.23) as

$$\mu_{\text{ENU}}^{\text{OPX}} = \underbrace{\mu_{\text{ENU}}^{\text{SCI}}}_{\substack{\text{independent}\\ \text{of planet}}} \mu_{\text{SCI}}^{\text{OPX}}\,, \tag{8.35}$$

and so use (8.28), (8.30), and (8.34) to write

$$\begin{aligned} \mu_{\text{ENU}}^{\text{SCI}} &= \mu_{\text{ENU}}^{\text{ECEF}}\, \mu_{\text{ECEF}}^{\text{ECI}}\, \mu_{\text{ECI}}^{\text{SCI}} \\ &= \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix} E_2^{\lambda}\, E_3^{-\phi} E_3^{-\gamma} E_1^{\tau}\, E_3^{p}\,. \end{aligned} \tag{8.36}$$

Also, from (8.15) (with subscripts $J$ and $E$ for Jupiter and Earth),

$$\mu_{\text{SCI}}^{\text{OPJ}} = E_3^{\Omega_J} E_1^{i_J}\, E_3^{\omega_J}\,, \qquad \mu_{\text{SCI}}^{\text{OPE}} = E_3^{\Omega_E} E_1^{i_E}\, E_3^{\omega_E}\,. \tag{8.37}$$

The sought-after coordinate vector of Jupiter relative to Adelaide in ENU coordinates $[\boldsymbol{r}_{JA}]_{\text{ENU}}$ can now be calculated from (8.16). Next, write the components of $[\boldsymbol{r}_{JA}]_{\text{ENU}}$ as $x_{\text{ENU}}, y_{\text{ENU}}, z_{\text{ENU}}$, and from these extract Jupiter's bearing $\beta$ (ground angle from north to east) and elevation $\varepsilon$ via

$$\sin\beta = x_{\text{ENU}}/D\,, \quad \cos\beta = y_{\text{ENU}}/D\,, \quad \tan\varepsilon = z_{\text{ENU}}/D\,, \tag{8.38}$$

where $D \equiv \sqrt{{x_{\text{ENU}}}^2 + {y_{\text{ENU}}}^2}$.

# 9 So Where is Jupiter?

We now combine everything with an example: in which direction must one look from an Adelaide back yard to find Jupiter at 9:00 p.m. local (daylight savings) time on 22nd March 2014?

Begin by collecting the orbital elements for Jupiter and Earth; these can be found on the Internet [28]. The elements used here apply to the J2000.0 epoch, and I have quoted a set that uses the "longitude of perihelion" and "mean longitude" just to show how to deal with these two strange and unnecessary parameters. Also, aside from the elements themselves, to save space I will write most numbers below to two significant decimal places, but will use more decimal places in the calculations.

Convert the requested date to GMT and then to a julian day, do the same for the epoch, and you'll find that the time elapsed since the epoch is 5193.9375 days: this is $t - t_0$ in (8.17), the duration for which we require to evolve the planet's position from its initial position that was given in its orbital elements. Do this by applying Kepler's equation to Jupiter and Earth in turn, as follows.

### The Calculation for Jupiter

Referring to (8.16), calculate $\mu_{\text{ENU}}^{\text{OPJ}}$ from (8.35) and $[\boldsymbol{r}_{JS}]_{\text{OPJ}}$ from Kepler' theory, (8.18). Jupiter's mass and orbital elements are

$$\begin{gathered} \text{mass} = 1.8986 \times 10^{27} \text{ kg}, \\ a = 5.203\,363\,01 \text{ AU}, \quad e = 0.048\,392\,66, \\ i = 1.305\,30^\circ, \qquad \Omega = 100.556\,15^\circ, \end{gathered}$$

$$\begin{aligned}\text{“long. of perihelion”} &= 14.753\,85°,\\ \text{“mean longitude”} &= 34.404\,38°.\end{aligned} \tag{9.1}$$

(The semi-major axis $a$ is given in "astronomical units", where 1 AU is Earth's mean distance from the sun, or $1.495\,978\,707 \times 10^{11}$ metres.) Without hesitation, apply (7.1) to decrypt and delete the last two quantities in (9.1), replacing them with the meaningful parameters

$$\omega = -85.80°, \quad M_0 = 19.65°. \tag{9.2}$$

We use a value for the gravitational constant of $G = 6.67384 \times 10^{-11}$ SI units. Convert the semi-major axis length $a$ to metres, then calculate the semi-minor axis length $b = a\sqrt{1-e^2}$. Calculate Jupiter's period $T$ from (3.3), using a solar mass of $M = 1.989 \times 10^{30}$ kg and Jupiter's mass $m$ in (9.1): the result is $T = 4332.8$ days.

Apply (8.17) to evolve Jupiter's mean anomaly at epoch, $M_0$, to its value at the requested time, $M$. (Don't confuse this $M$ with the Sun's mass.) The result is $M = 91.20°$. Solve $E - e\sin E = M$ for Jupiter's eccentric anomaly $E$ at the requested time: $E = 93.97°$ (use radians in the calculation!). Apply (8.18) to find Jupiter's position relative to the Sun in Jupiter's orbit-plane coordinates:

$$[\boldsymbol{r}_{JS}]_{\text{OPJ}} = \begin{bmatrix} -0.91 \\ 7.76 \\ 0 \end{bmatrix} \times 10^{11}\,\text{m}. \tag{9.3}$$

Now use (8.35)–(8.37) to calculate $\mu^{\text{OPJ}}_{\text{ENU}}$. We need:

$$\begin{gathered}\lambda = -34.9° \text{ [Adelaide's latitude]}, \qquad \phi = 138.60° \text{ [Adelaide's longitude]},\\ \gamma = 280.46° + 56.71° \text{ [Greenwich sidereal angle (6.2)]}, \qquad \tau = 23.439° \text{ [Earth's tilt]},\\ p = \frac{360° \times 5193.9375 \text{ days}}{25{,}770 \text{ years} \times 365.25 \text{ days/year}} = 0.20° \text{ [Earth's precession]}.\end{gathered} \tag{9.4}$$

These give

$$\mu^{\text{SCI}}_{\text{ENU}} \overset{(8.36)}{=\!=} \begin{bmatrix} -0.90 & -0.40 & 0.17 \\ -0.25 & 0.80 & 0.55 \\ -0.35 & 0.45 & -0.82 \end{bmatrix}, \quad \mu^{\text{OPJ}}_{\text{SCI}} \overset{(8.37)}{=\!=} \begin{bmatrix} 0.97 & -0.25 & 0.022 \\ 0.25 & 0.97 & 0.0042 \\ -0.023 & 0.0017 & 1.00 \end{bmatrix}. \tag{9.5}$$

While not necessary to the main calculation, the position of Jupiter relative to the Sun in SCI coordinates is

$$[\boldsymbol{r}_{JS}]_{\text{SCI}} = \mu^{\text{OPJ}}_{\text{SCI}}\,[\boldsymbol{r}_{JS}]_{\text{OPJ}} = \begin{bmatrix} -2.86 \\ 7.27 \\ 0.034 \end{bmatrix} \times 10^{11}\,\text{m}. \tag{9.6}$$

This coordinate vector suggests two short checks. First, the relative smallness of the third element agrees with the fact that Jupiter's orbit plane nearly coincides with that of Earth, since Earth's orbit plane defines SCI coordinates. Second, the length of $[\boldsymbol{r}_{JS}]_{\text{SCI}}$ is 5.2 AU, which is approximately equal to the value of $a$ in (9.1)—as expected, since Jupiter's orbit has a small eccentricity.

What we *do* need is the matrix $\mu^{\text{OPJ}}_{\text{ENU}}$ in (8.16): this is simply the product of the matrices in (9.5) in the order listed. We've finished with Jupiter, and are halfway through the main calculation.

### The Calculation for Earth

Repeat the above steps for Earth, using its mass and orbital elements

$$\begin{gathered}\text{mass} = 5.9736 \times 10^{24}\ \text{kg},\\ a = 1.000\,000\,11\ \text{AU}, \quad e = 0.016\,710\,22,\\ i = 0.000\,05°, \qquad \Omega = -11.260\,64°,\\ \text{“long. of perihelion”} = 102.947\,19°,\\ \text{“mean longitude”} = 100.464\,35°.\end{gathered} \tag{9.7}$$

Note that Earth's orbital inclination $i$ is not exactly zero here owing to astronomers using a mean plane in their measurements of Earth's orbit; this is a higher-order correction to our calculation that won't concern us. Astronomers give $\Omega$ a value for the same reason. The calculations in this report would ordinarily be insensitive to a value of $\Omega$ for Earth, but the one tabulated in (9.7) must be used to decrypt the "longitude of perihelion", using (7.1).

By following the same steps as for Jupiter, we arrive at the position of Earth relative to the Sun in Earth's orbit-plane coordinates:

$$[\boldsymbol{r}_{ES}]_{\text{OPE}} = \begin{bmatrix} 0.28 \\ 1.46 \\ 0 \end{bmatrix} \times 10^{11}\ \text{m}. \tag{9.8}$$

The calculation of $\mu^{\text{OPE}}_{\text{ENU}}$ proceeds in the same way as for Jupiter, producing

$$\mu^{\text{OPE}}_{\text{ENU}} = \begin{bmatrix} -0.18 & 0.97 & 0.17 \\ 0.83 & 0.060 & 0.55 \\ 0.52 & 0.24 & -0.82 \end{bmatrix}. \tag{9.9}$$

The calculation for Earth is finished. Next, (8.16) specifies $\mu^{\text{ECEF}}_{\text{ENU}}$ and $[\boldsymbol{r}_{AE}]_{\text{ECEF}}$ to be calculated.

### Adelaide's Position Relative to Earth's Centre

Converting between ECEF coordinates and the east–north–up coordinates centred on Adelaide is enabled via $\mu^{\text{ECEF}}_{\text{ENU}}$, given by (8.28):

$$\mu^{\text{ECEF}}_{\text{ENU}} = \begin{bmatrix} -0.66 & -0.75 & 0 \\ -0.43 & 0.38 & 0.82 \\ -0.62 & 0.54 & -0.57 \end{bmatrix}. \tag{9.10}$$

The final term necessary in (8.16) is the position of Adelaide relative to Earth's centre, in ECEF coordinates. Equation (8.22) gives

$$[\boldsymbol{r}_{AE}]_{\text{ECEF}} = \begin{bmatrix} -3.93 \\ 3.46 \\ -3.63 \end{bmatrix} \times 10^{6}\ \text{m}. \tag{9.11}$$

### Combining the Matrices and Coordinate Vectors

We now insert all necessary matrices and coordinate vectors into (8.16) to arrive at

$$[\boldsymbol{r}_{JA}]_{\text{ENU}} = \begin{bmatrix} -1.66 \\ 6.21 \\ 3.76 \end{bmatrix} \times 10^{11}\,\text{m} \equiv \begin{bmatrix} x_{\text{ENU}} \\ y_{\text{ENU}} \\ z_{\text{ENU}} \end{bmatrix}. \tag{9.12}$$

Apply (8.38) to $x_{\text{ENU}}, y_{\text{ENU}}, z_{\text{ENU}}$ to extract the direction of Jupiter in Adelaide's sky:

$$\text{bearing } \beta = 345.1°, \text{ elevation } \varepsilon = 30.34°. \tag{9.13}$$

Jupiter appears fairly low down, just west of north. Our result assumes no atmosphere, but it turns out that atmospheric refraction increases this particular elevation by only 0.03°.

We have assumed the orbital elements to be constant: $\Omega(t) = \Omega(t_0)$ etc. They can be evolved linearly over $t - t_0$ using their small tabulated rates of increase, which I have not included in this report. But this refinement turns out to be negligible for the short period of time over which we are evolving the orbit's parameters from the J2000.0 epoch.

The direction we have calculated differs by 0.15° from that returned by the excellent *Stellarium* software, which gives a bearing/elevation of 344.9°/30.31° for the case of no atmosphere. Stellarium adds empirical refinements to its calculations: it applies standard formulae to calculate parameters that do change slowly over long time scales, such as the length of the sidereal day. For simplicity, we have treated this length as constant.

For extra precision and to predict farther into the future, several other factors can be incorporated. The easiest is atmospheric refraction [29]. The travel time of light from Jupiter to Earth can also be considered—although this turns out to be negligible for the calculation above. (That Earth turns appreciably during the light-transit time is not relevant; the important point is that Jupiter doesn't move far along its orbit in that time.) Then there is Earth's changing orientation due to its non-trivial nutation, and the fact that its day is slowly lengthening due to the Moon's tidal drag. We could also include the ecliptic plane's small non-inertiality in the definition of the Sun-Centred Inertial frame SCI, and add various other terms of ever-decreasing size that relate to the way the Solar System evolves over time.

One can use the above theory to build a complex visual picture of our Solar System, by predicting the positions of planets as seen from any other planet: the calculation simply replaces Earth with the new home planet. From here it's only a short step to predicting the direction of, say, Jupiter's moon Io as seen in the sky at some specified point on Neptune's moon Nereid.

# 10 The Sun and Moon

The Sun's position as seen from, say, Adelaide can easily be predicted. The relevant vector is just the reverse of Adelaide's position with respect to the Sun: $\boldsymbol{r}_{SA} = -\boldsymbol{r}_{AS}$, and it can be coordinatised in whatever coordinate system is useful. In particular, calculating $[\boldsymbol{r}_{SA}]_{\text{ENU}}$ results in times of sunrise and sunset when you test trial times for when the Sun's elevation is zero—although atmospheric refraction is more significant then and should be included in the calculation.

What about predicting the Moon's position? Earth, Moon, and Sun form a genuine 3-body system [30], but this only really affects calculations of the Moon's position. Pinpointing that position accurately requires adding several empirical terms to the scheme described above. These terms are not described here but, even so, we can predict the position to an accuracy of a degree or so by treating the Moon and Earth as a 2-body system using the above analysis. (Unfortunately, because the Moon is half a degree wide, the result isn't accurate enough to predict eclipses.) The calculation is similar to that for Jupiter, but the 3-body nature of the Moon's orbit obliges us to use a measured value of its period rather than use (3.3) to calculate the period. The period is of course roughly one month, but several types of month are defined, and to use the above calculation of anomalies you must use the *anomalistic month*, the time between successive perigees, of 27.554550 days. Also, the two orbital elements $\Omega$ and $\omega$ are now no longer even remotely constant. The Moon's ascending node regresses within the ecliptic plane (i.e. moves left handed about $z_{\text{SCI}}$) through 360° every 18.61 years, so that $\mathrm{d}\Omega/\mathrm{d}t = -360°/(18.61 \text{ years})$. If you stand on this moving node and watch the Moon's orbit plane, you'll notice that the Moon's perigee rotates within that orbit plane through 360° every 6.00 years in the same direction that the Moon orbits, resulting in $\mathrm{d}\omega/\mathrm{d}t = 360°/(6.00 \text{ years})$ [31]. These two rates of increase are easily used to evolve $\Omega$ and $\omega$ from the epoch to the requested time. Then just apply (8.15) as usual, using $\Omega(t)$ and $\omega(t)$ instead of $\Omega(t_0)$ and $\omega(t_0)$.

# 11 Finding Stars and Nebulae

Finally, it becomes a simple matter to use a subset of the above analysis to calculate the bearing and elevation of any "fixed" celestial object seen from any place and time on Earth. Stellar positions are tabulated as the celestial longitude and latitude of their directions from Earth with reference to Earth-Centred Inertial coordinates: that is, the sight direction's angle east of the First Point of Aries in Earth's equatorial plane $x_{\text{ECI}}\, y_{\text{ECI}}$, and its angle north of Earth's equatorial plane. (The extension of Earth's equatorial plane out to a great circle in the sky produces the *celestial equator*, and the vanishing points of Earth's rotation axis in the sky are the *celestial poles*.) Astronomers call an object's celestial longitude its *right ascension*, and the celestial latitude its *declination*. We can also calculate right ascension and declination of the planets using the analysis above, and so plot their positions in a star atlas.

# 12 How Far Ahead Can We Predict?

If we use the above numbers and 2-body calculation to predict Jupiter's position for some date in AD 3000, our result will differ from that of Stellarium by a dozen degrees. Clearly we need more knowledge of the various changing parameters to predict 1000 years into the future. Evolving the orbital elements using their first time derivatives reduces this error to about 7°. In fact, these orbital elements and their rates of increase are determined partly from observation, and partly from running computer simulations of planets' orbits and least-squares fitting the elements and their rates of increase to the output; so from our point of view the exercise of increasing our precision becomes a little artificial.

At the end of the day, the hunt for more precision is not a question of requiring more physics. We have all the physics we need, but modelling an $n$-body system by pairs of 2-body

solutions will only ever be an approximation. We cannot hope to predict arbitrarily far ahead with such a model, unless we're prepared to add an arbitrarily large number of empirical terms that modify those 2-body solutions to fit the true $n$-body solution And this $n$-body solution is only the result of solving Newton's gravity numerically in some approximation to that most elusive of constructs: the long-lived inertial frame.

# 13 Concluding Remarks

If you have followed the calculations in this report, you'll have gained more than you might realise. You will have a better understanding of Earth's place in our Solar System. You'll be able to tackle the mostly arcane books on orbital theory, which tend to derive much of the above theory—when they derive it at all—in more convoluted ways than I have done here. The $\mu$ matrices of Section 8 form the absolute core of orientational analysis, so you will have a better understanding of literature such as used in modern 3D movie making and aerospace 6 degree-of-freedom modelling [32]. With some computer graphics skills you could write your own planetarium software. And you'll gain an appreciation for the efforts of early astronomers, who helped create our modern world without the benefit of our modern mathematical and computational tools.

# 14 References

[1] For an example of this, see P. Duffett-Smith, J. Zwart (2011), *Practical Astronomy with your Calculator or Spreadsheet*, 4$^{\text{th}}$ ed., Cambridge University Press. Modelling the Sun as orbiting Earth is useful for qualitative discussions, but for serious calculation, such a model only obscures the essential simplicity of the subject, both physically and mathematically. It tends to read like a mediaeval manuscript.

[2] One well-known book that is essentially a collection of recipes is J. Meeus, (2009), *Astronomical Algorithms*, 2nd ed., Willmann–Bell. A short book that is easy to read but difficult to use practically is R. Madonna, (1997), *Orbital Mechanics*, Krieger Publishing. A book that has good content but without giving real insight (and does not use matrices) is O. Montenbruck, T. Pfleger, (1989), *Astronomy on the Personal Computer*, Springer. A readable book that also doesn't use matrices and gives no major worked example is P. van de Kamp, (1964) *Elements of Astromechanics*, W.H. Freeman. Other books have a huge amount of information, but not always what the time-poor reader necessarily wants to have to work through on a first reading. Examples are D. Vallado, (2007), *Fundamentals of Astrodynamics and Applications*, 3rd ed., Space Technology Library; P. Escobal, (1965), *Methods of Orbit Determination*, John Wiley; J. Vinti, (1998), *Orbital and Celestial Mechanics*, AIAA Education; F. Moulton, (1914), *An Introduction to Celestial Mechanics*, Macmillan Group. A book with a lot of theory even in its introduction that would be avoided by most would-be practitioners, at least on a first read, is P. Fitzpatrick, (1970), *Principles of Celestial Mechanics*, Academic Press.

[3] P. Zipfel, (2014), *Modeling and Simulation of Aerospace Vehicle Dynamics*, 3rd ed., AIAA Education

[4] Refer to (2.2) if you have cause to wonder why a planet's "reduced mass" is defined to be $Mm/(M+m)$.

[5] See reference [25] for the report that explains these concepts in detail.

[6] This works because the basis vectors of cartesian coordinates don't depend on position, from which it follows that in the inertial frame in which we are calculating, they don't change with time. In such a case with cartesian coordinates $A$, any proper vector $\boldsymbol{v}$ obeys $[\mathrm{d}\boldsymbol{v}/\mathrm{d}t]_A = \mathrm{d}[\boldsymbol{v}]_A/\mathrm{d}t$.

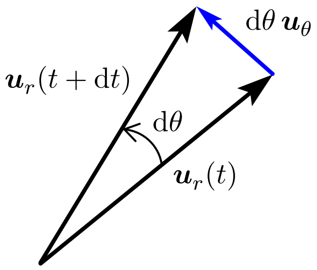
[7] Equation (2.7) can be derived without using the cartesian coordinate approach of (2.4)–(2.6). Start by differentiating $\boldsymbol{r} = r\boldsymbol{u}_r$ once:

$$\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{u}_r + r\dot{\boldsymbol{u}}_r\,. \tag{14.1}$$

What is $\dot{\boldsymbol{u}}_r$ in terms of $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$? Write the derivative using infinitesimal notation as

$$\dot{\boldsymbol{u}}_r = \frac{\boldsymbol{u}_r(t+\mathrm{d}t) - \boldsymbol{u}_r(t)}{\mathrm{d}t}\,. \tag{14.2}$$

Now consider $\boldsymbol{u}_r$ "pinned" to the planet, and ask how it evolves as the planet moves from time $t$ to $t+\mathrm{d}t$. Changing $r$ has no effect on $\boldsymbol{u}_r$, so consider in the following figure only what happens to $\boldsymbol{u}_r$ as $\theta$ changes:




In a time $\mathrm{d}t$, $\theta$ increases by $\mathrm{d}\theta$, causing $\boldsymbol{u}_r$ to turn through $\mathrm{d}\theta$. Thus the increment $\boldsymbol{u}_r(t+\mathrm{d}t) - \boldsymbol{u}_r(t)$ has length $\mathrm{d}\theta$ and, in this limit of infinitesimal rotation, is rotated 90° from $\boldsymbol{u}_r$; so this increment vector is just $\mathrm{d}\theta\,\boldsymbol{u}_\theta$. Hence (14.2) becomes $\dot{\boldsymbol{u}}_r = \mathrm{d}\theta\,\boldsymbol{u}_\theta/\mathrm{d}t = \dot\theta\boldsymbol{u}_\theta$, so that (14.1) is $\dot{\boldsymbol{r}} = \dot r\boldsymbol{u}_r + r\dot\theta\boldsymbol{u}_\theta$. Now repeat this idea: differentiate this last equation with respect to time, and now calculate $\dot{\boldsymbol{u}}_\theta$ similarly to (14.2) to give $\dot{\boldsymbol{u}}_\theta = -\dot\theta\boldsymbol{u}_r$. We arrive at the sought-after expression for $\ddot{\boldsymbol{r}}$ in terms of $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$, which is equation (2.7) once again:

$$\ddot{\boldsymbol{r}} = (\ddot r - r\dot\theta^2)\boldsymbol{u}_r + (2\dot r\dot\theta + r\ddot\theta)\boldsymbol{u}_\theta\,. \tag{14.3}$$

[8] For an alternative to this vector-calculus approach of producing (2.8), try the approach of writing Lagrange's equations in the variables $r$ and $\theta$, as Lagrange's approach doesn't rely on vectors.

[9] F. van Diggelen (2009) *A-GPS: Assisted GPS, GNSS, and SBAS*, Artech House. See Chapter 8, *Ephemeris Extension, Long-Term Orbits*.

[10] J. Connor (2005), *Kepler's Witch*, HarperOne. Page 5 cites an essay by eminent science historian I. Bernard Cohen.

[11] You'll sometimes encounter the idea that a sort of fiducial idealised satellite can be envisaged as moving in a circular orbit at constant speed about the Sun, with an angle from perihelion given at any moment by the mean anomaly. This picture is just that, but has no physical or mathematical utility.

[12] When using trigonometric functions to specify an angle implicitly, we must *always* specify two pieces of information about that angle: say, its sine and cosine, or any one of its sine, cosine, or tangent along with its quadrant, etc. Most computer languages have a function `atan2` that returns $\theta$ when given $\sin\theta$ and $\cos\theta$ as arguments (and be sure always to check the order that these arguments must have). In particular, it's incorrect to think that (4.11) can be expressed in the form "$\theta = \tan^{-1}\frac{\sin\theta}{\cos\theta}$" for all $\theta$—a wrong piece of mathematics that you'll often find in books and journal papers. The inverse tangent function simply isn't defined that way; and making such a mistake has the effect of discarding one of the two required pieces of information. No inverse tangent function can be defined in such a way, because any function that returns $\theta$ *requires* two pieces of information. The correct expression is "$\theta = \tan^{-1}\frac{\sin\theta}{\cos\theta}$ $(+\,\pi$ if $\theta$ is in quadrant 2 or 3)".

[13] I won't complicate things by saying that the julian day's GMT differs from "modern GMT" (Coordinated Universal Time, UTC) by some leap seconds, which need to be carefully accounted for in high-precision calculations. Several other time scales are also defined in this subject. For more information, see `http://www.bipm.org/en/bipm-services/timescales`.

[14] Bizarrely and deplorably, the terms julian day and julian date have in recent years sometimes been misappropriated as names for the *ordinal date*, which is simply a number for the current day of the year, from 1 to 365 or 366, which resets to 1 at the start of each year.

[15] I emphasise that 12:00 noon GMT on 1st January 4713 BC in the proleptic julian calendar is the *same moment* as 12:00 noon GMT on 24th November 4714 BC in the gregorian calendar.

[16] Frames and coordinates are different things. A frame is physical: a choice of viewpoint visualised as a lattice within which we view and describe the physics of a situation. Coordinates are mathematical: sets of numbers that locate events relative to the lattice, and used to perform calculations. A given frame can be described by any number of different coordinate systems (e.g. cartesian and polar); and conversely, a given coordinate system can be used in any number of different frames (e.g. cartesian is typically used for SCI and ECI).

[17] The word *solstice* means, in essence, "Sun stationary", denoting the two moments in the year when the Sun reaches its most northerly and southerly points in the sky. The word *equinox* means "equal night", referring to the fact that around the equinox as the Sun crosses the celestial equator, day and night have approximately equal durations. These two durations can only be approximately equal because the Sun doesn't necessarily cross that equator at a convenient civil time.

[18] The First Point of Aries is often called the vernal equinox or spring equinox, both terms that I see as misleading and should not be used. A point in the sky and a moment in time are two separate things, and we need terms for both: the First Point of Aries is unambiguously a point in the sky, so the equinox ("equal night") should be reserved for a moment in time (when the Sun crosses the celestial equator). "Vernal" means green, referring to the season of spring which begins in the northern hemisphere around March, and this certainly refers to time; however, what astronomers often call the "spring equinox" is a moment in time that occurs in the southern hemisphere's autumn. No confusion will ever arise if you refer to the point in the sky as the First Point of Aries, and the moment in time as the March equinox (as well as the June solstice, September equinox, and December solstice).

[19] Why does Earth precess? Earth's equatorial bulge feels unbalanced pulls in the inhomogeneous gravity fields of both Sun and Moon. The resulting torque causes Earth to precess as it spins. In contrast to a spinning toy top that is trying to be toppled by the combination of gravity and the contact force of the table it sits on, Earth is trying to be "righted" by Sun and Moon, so it precesses in the *opposite* direction to that of a spinning top. Earth's bulge, along with displaced water in its tides that drag along Earth's surface slightly out of phase with the Moon that causes them, gives Earth a slightly non-radial gravitational field that applies a tiny sideways force to the Moon, which doesn't quite orbit in Earth's equatorial plane. This tiny force "micro sling-shots" the Moon away from Earth, resulting in the Moon's distance from us increasing by several centimetres per year.

[20] US Naval Observatory Circular No. 179 (2005), freely available from their web site.

[21] Besides being used as a measure of angle by astronomers, the sidereal hour is sometimes said to denote a measure of time relative to the stars instead of the Sun: 24 sidereal hours is said to equal one sidereal day, or about 23.9345 civil clock hours. Those who insist on this usage now have *two* kinds of temporal hour and one angular hour. When the angular hours are subdivided into minutes and seconds, these latter are not even remotely related to the commonly used minutes and seconds of arc. Forty years on from first encountering these astronomical concepts, I have yet to find any use for such artificial complexity in what is really a straightforward subject.

[22] What comes across as a contrariness in the continued usage of these two sums seems to be part of the culture of modern astronomy. I have already mentioned the name "sidereal time" for angle, along with its two definitions of "hour" that are often both used in the same sentence. But the list goes on. Some (not all) planetary specialists define longitude as positive *west* of a planet's prime meridian, against all mathematical and geophysical usage. And astronomers who specialise in measuring radiation routinely plot a hot emitter's "power radiated per unit emitter surface area per unit frequency" versus not *frequency*, but *wavelength*—which is not mathematically wrong per se, but which does produce plots that cannot be interpreted in the intuitive way of true density plots. This practice requires the usual textbook maths of blackbody radiation to be reformulated (such as Wien's Law), to incorporate what is really a misuse of the concept of density.

Witness the field's widespread and lone use of the "erg", a unit of energy that equals $10^{-7}$ joules, when it would surely be far more appropriate to use a *large* unit, such as $10^{+24}$ joules, when speaking of exploding stars. Similarly, instead of measuring distances in light years that everyone understands, some astronomers insist on using the *parsec* (short for "parallax-second"), the distance at which a star has a half-yearly parallax of one arc second when seen from Earth orbiting the Sun. The parsec dates from the dawn of stellar distance measurement, but its definition has no modern use for anything much farther than a half dozen of the nearest stars; and given that one parsec equals about 3.26 light years, the unit

doesn't even introduce any economy of numerical use as compared with the light year. Quoting distances in parsecs nowadays comes across as an exercise in deliberate obscurity.

[23] The distinction between proper vectors and coordinate vectors, along with equation (8.6) and the pseudo-reversing theorem discussed after (8.10), form the heart of orientation/rotation theory. If you digest what I have written, you'll be able to perform and understand orientational calculations very easily.

[24] Canonical pictures of such rotation sequences sometimes appear in books, but I haven't provided any because you are probably better off visualising my description rather than attempting to work out what is going on in those pictures.

[25] Very few books mention the pseudo-reversing theorem, but try proving it by first twisting your hand, then rotating your body; now return to the start position and rotate your body, then twist your hand. Ask yourself precisely which axes these rotations are being done around, and establish that any number of rotations can be "reversed" in this way when we alter the sense of the rotation axes appropriately. You can find this method of reversing the rotation order explained in great detail in D. Koks (2012), A Pseudo-Reversing Theorem for Rotation and its Application to Orientation Theory, DSTO–TR–2675, Melbourne, Vic., Defence Science and Technology Organisation (Australia).

[26] Some books change the sign of $\theta$ in their Euler matrices because they define those matrices with an opposite sense of rotation to that used in this report. Additionally, notice that e.g. $E_1^\theta$ serves to rotate a vector around *any* generic $x$ axis. For example, when a vector $\boldsymbol{v}$ is rotated through $\theta$ around the $x_A$ axis of coordinate system $A$, the result is $\boldsymbol{v}'$, where

$$[\boldsymbol{v}']_A = E_1^\theta \, [\boldsymbol{v}]_A \,,$$

and when $\boldsymbol{v}$ is rotated through $\theta$ around the $x_B$ axis of coordinate system $B$, the result is $\boldsymbol{v}''$, where

$$[\boldsymbol{v}'']_B = E_1^\theta \, [\boldsymbol{v}]_B \,.$$

Hence we don't write $E_1^\theta$ as "$E_x^\theta$" because there may well be an $x$ axis present, and the action of the matrix is not confined to rotation about only that axis.

[27] Bearing and elevation are often called "azimuth" and "altitude" respectively in astronomy. Outside that subject, "azimuth" doesn't necessarily carry a sense of being referred to north, and of course "altitude" more usually denotes a height, not an angle.

[28] I have used those at http://www.met.rdg.ac.uk/~ross/Astronomy/Planets.html

[29] J. Meeus (1998), *Astronomical Algorithms*, 2nd ed., Willmann–Bell. Equation (16.4).

[30] Earth and Moon are actually better regarded as a double planet. Our Moon has exceptional orbital and physical features that distinguish it over the other satellites in the Solar System, blurring its orbital character to an extent that means it can't simply be treated as Earth's satellite. The Moon really orbits the Sun in step with Earth, so that calculations of its orbit must take Earth's motion about the Sun into consideration. Strictly speaking, a "keplerian orbit" relates only to the 2-body problem; but unlike other satellites in the Solar System, the Moon's motion is well and truly that of a 3-body problem, for which no analytical solutions are known. Incidentally, our Moon probably has more in common with the major planets than the planet Pluto lacks, which shows the logical inconsistency displayed by the mostly non-planetary specialists of the International Astronomical Union, when a minority of members decreed in 2006 that we must all no longer call Pluto a planet.

[31] The value of 6.00 years is well known. Don't use the figure of 8.85 years found on many web sites. The 8.85 years actually relates an anomalistic month to a "sidereal month", which we are not using.

[32] For example, differentiate (8.5) with respect to time to produce what is called the *rotational derivative* in the reference that follows, a comparatively recent concept in aerospace literature that is actually the aerospace version of the *covariant derivative* of vector and tensor calculus. For an introduction to the rotational derivative, see Zipfel [3]. That reference writes our $[\boldsymbol{v}]_A = \mu_A^B \, [\boldsymbol{v}]_B$ as $[v]^A = [T]^{AB}[v]^B$. (Additionally, the use in [3] of "$[\mathrm{d}s/\mathrm{d}t]^A$" in an arbitrary frame [not necessarily inertial] must be interpreted in that book's notation as really $\mathrm{d}[s]^A/\mathrm{d}t$ to be mathematically meaningful [contrast with the comment in [6], which assumes inertiality]. Also, its theory of the rotational derivative can be simplified notationally,

but is still accessible.) Note that the $\mu$ matrices in this report are partly my own notation; you'll find all manner of other notation in aerospace literature. Most of that literature doesn't distinguish between proper vectors (arrows) and coordinate vectors (arrays of numbers)—a distinction that I see as crucial for an in-depth understanding of calculating with vectors in multiple coordinate systems.

# Index

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19. ABSTRACT

This note works through an example of switching between many coordinate systems using a modern matrix language that lends itself to describing arenas with multiple entities such as found in many Defence scenarios. To this end, it describes an example in planetary orbital theory, whose various Sun- and Earth-centred coordinate systems makes that theory a good test-bed for such an exposition of changing coordinates. In particular, we predict the look direction to Jupiter from a given place on Earth at a given time, highlighting the careful book-keeping that is required along the way. To avoid much of the rather antiquated jargon and notation that pervades orbital theory, we explain the first principles of 2-body orbital motion (Kepler's theory), beginning with Newton's laws and proving all the necessary expressions. The systematic and modern approach to changing coordinates described here can also be applied just as readily in contexts such as a Defence aerospace engagement, which follows the interaction of multiple entities that each carry their own coordinate system.